

THE STRUCTURE OF EXTENDED BODIES IN GENERAL RELATIVITY

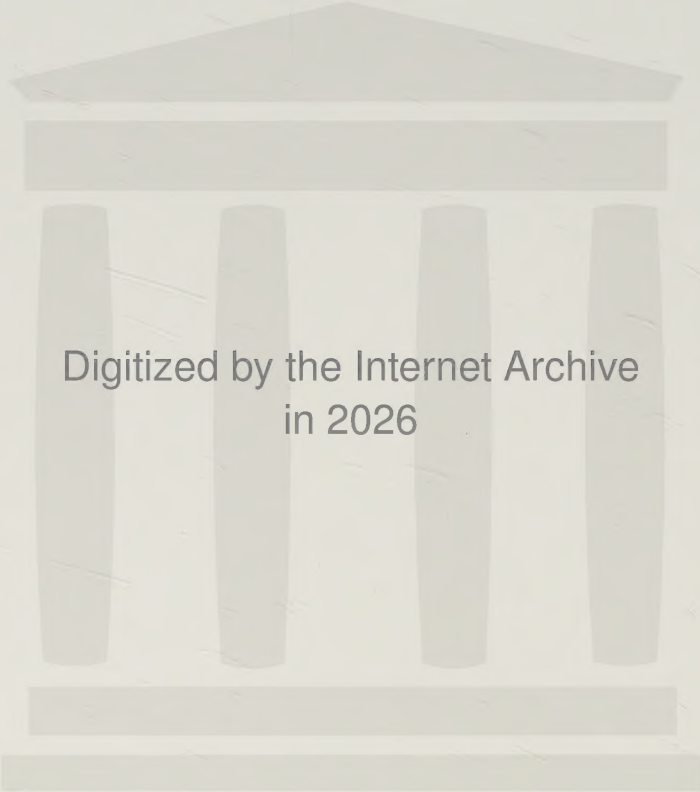
(Die Struktur ausgedehnter Körper in der Allgemeinen
Relativitätstheorie)

Dissertation der Fakultät für Physik der Ludwig-Maximilians-
Universität München

vorgelegt von
Ruprecht H. Schattner
aus —
Eichstätt

München, den 11.12.1979

1980
—



Digitized by the Internet Archive
in 2026

https://archive.org/details/IA41548319_0049

THE STRUCTURE OF EXTENDED BODIES IN GENERAL RELATIVITY
(Die Struktur ausgedehnter Körper in der Allgemeinen
Relativitätstheorie)

Dissertation der Fakultät für Physik der Ludwig-Maximilians-
Universität München

vorgelegt von
Ruprecht H. Schattner
aus
Eichstätt

München, den 11.12.1979

1. Gutachter: Prof. Dr. J. Ehlers
2. Gutachter: Prof. Dr. H. J. Meister
3. Gutachter: Prof. Dr. H. Dehnen

Tag der mündlichen Prüfung: 29. 7.1980



1. Die Bedeutung der Kunst

Die Kunst ist eine der wichtigsten Säulen der menschlichen Kultur. Sie spiegelt die Gedanken, Gefühle und Erfahrungen eines Volkes wider und trägt zur Entwicklung der menschlichen Seele bei. Durch die Kunst können wir uns mit anderen Menschen verbinden und eine gemeinsame Sprache finden. Die Kunst ist also nicht nur ein Hobby, sondern eine ernsthafte Aufgabe, die uns dazu zwingt, die Welt um uns herum zu betrachten und zu verstehen.

Meinen Eltern gewidmet

ὁ μὴ δαρὲς ἄνθρωπος οὐ παιδεύεται

ABSTRACT

In Dixon's theory of the dynamics of extended bodies in metric theories of gravity the relation between the energy-momentum tensor and the energy-momentum skeleton is analysed and generalized to distributional energy-momentum tensors. A class of distributions admissible as energy-momentum tensors is defined, the skeleton is shown to be a distribution in a certain class and a reconstruction-theorem is proven. Furthermore we inquire about the restrictions imposed on distributional energy-momentum tensors satisfying an energy-condition.

Contents

- O. Preliminaries
 - O.1 Introduction
 - O.2 Notation and Conventions
 - O.3 Schematic Diagram
- 1. Distributions and Test Functions on Dual Set Systems
 - 1.1 Dual Set Systems
 - 1.2 Locally Convex Topological Vector Spaces - Review of Definitions and Theorems
 - 1.3 Test Field Spaces on K-Families
 - 1.4 Distributions Based on K-Families
 - 1.5 Some Remarks on K-Family Based Distributions on Submanifolds which are the Closure of Open Subsets of the Manifold M
- 2. The Energy Momentum Distribution, the Test Field Spaces and Their Decomposition
 - 2.1 Geometry, the Basic K-Family and the Basic Function Space
 - 2.2 Auxiliary Test Field Spaces
 - Appendix A: The Most Important Bi-Tensor Fields
 - 2.3 Composable Families of Distributions
 - 2.4 The Decomposition of $(T_{[2]}^0, T)_M$
 - 2.5 Decomposition of $\mathcal{T}_{[2]}^0$ with Respect to a Timelike Curve Contained in W
 - Appendix A: Derivation of a Bi-Tensor Formula
 - Appendix B: Solution of the Inhomogeneous Jacobi Equation

3. Momentum, Angular Momentum and Skeleton of a Divergence-Free Distribution in $(\mathcal{F}_{[2]}^0)'$

3.1 Definitions

3.2 Properties of the Skeleton

Appendix A: Extended Skeleton, Moment Generating Function, Moments

3.3 Dixon's Integral Laws of Motion

3.4 Reconstruction of a Distribution from a Skeleton

4. Applications of the Developed Calculus

4.1 Simple Multipole Particles

4.2 The Dominant Energy Condition

5.1 List of Properties Occurring in Certain Definitions

5.2 Bibliography

O. PRELIMINARIES

O.1. Introduction

Since the comparison of general relativity theory with astrophysical observation is based mainly on measurements performed on practically isolated systems consisting of a finite number of well defined bodies, the study of such systems merits considerable attention. The interest in a theory treating the evolution of such a system has even increased since the report of J.H. Taylor, L.A. Fowler and P. McCulloch [27] on their measurements of the period change of the binary pulsar PSR 1913+16 given at the 9th Texas Symposium in Munich.

At present we are still far from exact quantitative statements about the motion of the constituents of such systems (apart from the 1-body case), nor do there exist reliable approximation methods, as has been pointed out e.g. by J. Ehlers, A. Rosenblum, J.N. Goldberg and P. Havas [12]. So far research on this subject has proceeded along three major lines: the first one deals with weak field approximations and equations of motion. For a survey see J. Ehlers [14]. The other two treat certain aspects of the problems in an exact, covariant manner: the asymptotic structure of spacetime far away from material sources and the local structure and motion of bodies relatively to an unspecified gravitational field. For an account on the asymptotics see, for example, F.P. Esposito and L. Witten (eds.) [16], J. Ehlers (ed.) [15],

S.W. Hawking and G.F.R. Ellis [18] and M. Streubel [24]. The local theory has been developped by W.G. Dixon [6, ..., 10] and continued by J. Ehlers and E. Rudolph [13] and R. Schattner [20, 21].

The aim of this work is to extend the local theory and, above all, to provide deeper understanding of its structure, in order to make possible or at least less deficient a successful attack on three major problems of the theory: how to connect the local quantities (as mass, momentum, angular momentum) with the corresponding asymptotic quantities, how to link Dixon's theory to approximation methods and how to attack the self field problem?

In the following the main subject of research is the relation between the energy-momentum skeleton, a mathematical object representing the structure of the "bodies", and the energy-momentum tensor. Dixon [9] has proven that for any smooth symmetric $(\overset{2}{0})$ tensor field T satisfying the local law of motion $\nabla \cdot T = 0$ whose support is contained in a spatially bounded, timelike worldtube W and for any timelike worldline $\mathcal{L}$ and any timelike unit vector field u along $\mathcal{L}$ there exist a momentum P , an angular momentum S and a skeleton $\hat{T}$ satisfying Dixon's integral laws of motion (and some further conditions in order to ensure the uniqueness of $\hat{T}$). P , S and $\hat{T}$ together, satisfying Dixon's integral laws of motion, contain the same information as an energy-momentum tensor T satisfying the local law of motion. The representation of T by $(P, S, \hat{T})$

has the decisive advantage that the complicated system of partial differential equations $\nabla \cdot T = 0$ has been replaced by a much simpler system of finitely many ordinary differential equations, Dixon's integral laws of motion. In fact this equivalence is the real justification for $\nabla \cdot T = 0$ being called "law of motion". The momentum P is a vector field along $\mathbb{L}$, the angular momentum S a bivector field along $\mathbb{L}$ and the skeleton $\hat{T}$ is a map which maps points of the spacetime manifold M into distributions on the space of symmetric $\binom{0}{2}$ tensor fields over the natural projection (for a definition, see section 2.2). Dixon's proof is constructive and Dixon gives a formula which indicates how to reconstruct $\hat{T}$ from $(P, S, \hat{T})$, but the following two problems remain open:

- (i) The proof shows existence and uniqueness of a linear functional $\hat{T}$ for given smooth T . But $\hat{T}$ is claimed to be and has to be a distribution, i.e. a continuous linear functional, since one applies to $\hat{T}$ operations (for example derivation, Fourier transform etc.) that are well defined only for distributions and not for general linear functionals. Therefore one has to show that $\hat{T}$ is continuous with respect to a suitable topology.
- (ii) Inserting an arbitrary triple $(P, S, \hat{T})$ in the reconstruction formula one obtains in general not a smooth tensor field T but a linear functional. So the question arises what spaces of T 's and $\hat{T}$'s one has to choose in order to make the relation between T and $(P, S, \hat{T})$ symmetric.

In order to approach these problems in a systematic way we develop at first a theory of spaces of distributions with certain support properties: In this behalf we introduce the notion of K-families, which are a natural generalization of both the family of closed subsets and the family of compact subsets of a manifold. Next we show that the vector space of smooth tensor fields with support in the elements of a K-family can be given a natural topology. The space of (tensor-) distributions based on the dual K-family of this K-family is then defined to be the topological dual of the above defined topological vector space.

In the second chapter the basic K-family (geometrical interpretation: the set of worldtubes describing the motion of finite bodies), the basic function and distribution spaces (the space containing the energy-momentum tensor) and certain auxiliary test field spaces are defined. Furthermore we introduce the notion of composable families of distributions and derive decomposition theorems for the test field spaces. These decomposition theorems are in their original form (without consideration of topological and support properties) due to Dixon and play the key role in the construction of the skeleton.

In the third chapter we define momentum P and angular momentum S for a distribution T and show that a "reasonable" skeleton $\hat{T}$ can be defined provided the energy-momentum distribution T satisfies certain conditions. In this case momentum, angular momentum and skeleton have the same properties as in the case

of a smooth energy-momentum tensor field, particularly they satisfy Dixon's integral laws of motion and the reconstruction equation. Finally, we show that for any triple $(P, S, \hat{T})$ with these properties the reconstruction equation defines a distribution T satisfying the above conditions, which symmetrizes completely the relation inbetween T and $(P, S, \hat{T})$. In the last chapter we use the developed calculus in order to prove some theorems on (distributional) energy-momentum tensors: We derive the energy-momentum tensor of a quadrupole and a monopole particle and the corresponding integral laws of motion, inquire about the consequences of the dominant energy condition and give a very general characterization of a monopole particle.

0.2. Notation and Conventions

Spacetime is described by a 4-dimensional C^∞ manifold M with C^∞ Lorentz metric $g = -\omega^1 \otimes \omega^1 - \omega^2 \otimes \omega^2 - \omega^3 \otimes \omega^3 + \omega^4 \otimes \omega^4$ in an orthonormal frame $\{\omega^i\}_{i=1..4}$. We assume M to be temporally and spatially oriented. Covariant differentiation (with respect to the Levi-Civita connection of g , unless otherwise stated) is denoted by ∇ , in local coordinate systems by ∇_a or ${}_{;a}$, partial differentiation by ∂ , ∂_a or ${}_{,a}$. Absolute differentiation along a curve $x(u)$ is denoted by D_u or D/du . $X \otimes Y$ is the symmetrized tensor product of X and Y , i.e. $X \otimes Y = \frac{1}{2} (X \otimes Y + Y \otimes X)$ for $X, Y \in TM$. In index notation we write $T_{(a_1 \dots a_n)}$ and $T_{[a_1 \dots a_n]}$ to denote symmetrization and antisymmetrization. Furthermore $T_{\{abc\}} := T_{abc} - T_{bca} + T_{cab}$, i.e. $\Gamma_{abc} = \frac{1}{2} g_{\{ab,c\}}$. The subspaces of $T_S^r M$ satisfying certain symmetry conditions are denoted by an indication of the

symmetry class in square brackets: For example the space of symmetric (antisymmetric) contravariant tensor fields of rank 2 corresponds to the irreducible symmetry $[2]$ (resp. $[1,1]$) and is therefore denoted by $T_{\circ}^{[2]} M$ (resp. $T_{\circ}^{[1,1]} M$). The decomposition $[1] \otimes [1] = [2] \oplus [1,1]$ corresponds to $T_{\circ}^2 M = T_{\circ}^{[2]} M \oplus T_{\circ}^{[1,1]} M$, i.e. $T^{ab} = T^{(ab)} + T^{[ab]}$. We define the differential operator $\mathfrak{f}$: $T_{[r]}^{\circ} M \rightarrow T_{[r+1]}^{\circ} M$ to be the adjoint of the divergence operator applied to $T_{\circ}^{[r+1]} M$, i.e. $\langle \nabla \cdot T, \lambda \rangle = \langle T, \mathfrak{f} \lambda \rangle$, and in local coordinates $\mathfrak{f}_{a_1} \lambda_{a_2 \dots a_r} = -\nabla_{(a_1} \lambda_{a_2 \dots a_r)}$. The sign of the curvature tensor is such that $A_{a;bc} - A_{a;cb} = R_{bca}^d A_d$, the Ricci tensor is given by $R_{ab} = R_{abc}^c$. η denotes the completely anti-symmetric tensor (volume form) satisfying $\eta_{1234} = 1$ in an oriented orthonormal tetrad. Sometimes we shall use the following letter-saving notation introduced in [13]

$$T_{abcd} U^b V^c = T_{aUVd} = T_a^{UV}{}_d.$$

Proposition 2.1: M admits a nowhere zero timelike vector field of class C^∞ .

Proof: From the existence of a C^∞ Lorentz metric we infer that M is paracompact and admits a timelike C^∞ line element field [18], that is an assignment of a pair of equal and opposite timelike vectors $(X, -X)$ to each $x \in M$. Using the time orientation we pick everywhere the future pointing vector. $\square$

Lemma 2.2: Let q be a timelike future directed (t.f.d.) C^∞ unit vector field on M . $e = 2q \otimes q - g$ is a positive definite (Riemannian) metric on M . $\square$

def. 2.1: Let e be a Riemannian metric on M , $x \in M$, $T \in \bigotimes_{s^M_x}^r$.

Then

$$|T|_e := \left[e_{a_1 b_1} \dots e_{a_r b_r} \bar{e}^{1 c_1 d_1} \dots \bar{e}^{1 c_s d_s} T_{c_1}^{a_1} \dots T_{c_s}^{a_r} T_{d_1}^{b_1} \dots T_{d_s}^{b_r} \right]^{1/2}$$

Lemma 2.3: $| \cdot |_e$ is a norm on $\bigotimes_{s^M_x}^r$. If $x \in M_x$ and $\omega \in M_x^*$, then

$$|\omega(x)| \leq |\omega|_e |x|_e$$

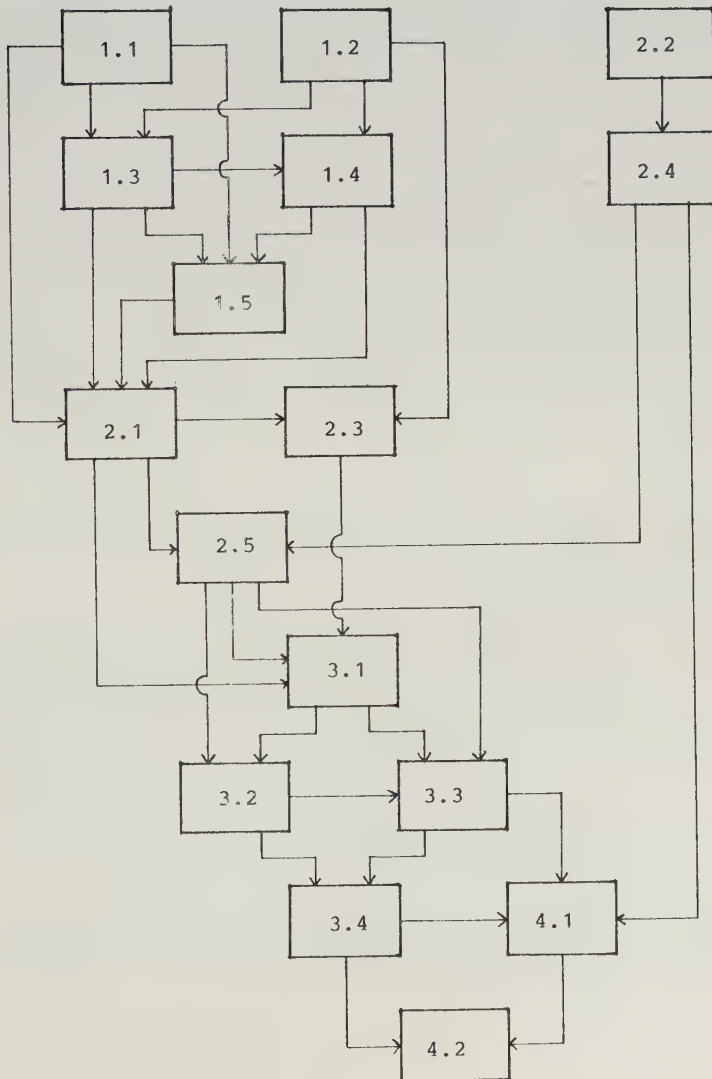
If $T \in \bigotimes_{s^M_x}^r$ and $S \in \bigotimes_{u^M_x}^t$ then

$$|T \otimes S|_e = |T|_e |S|_e$$

□

O.3. Schematic Diagram

The numbers in the blocks represent the sections, the arrows the logical hierarchy:



1. DISTRIBUTIONS AND TESTFUNCTIONS ON DUAL SET SYSTEMS

1.1. Dual Set Systems

In this section we introduce the concept of a K-family which is in some sense a generalization of the families of compact subsets of a manifold. We need these K-families in order to define spaces of distributions with certain support properties. The results of this chapter hold for any paracompact non-compact manifold M .¹⁾

def. 1.1: Let $\mathcal{O}$ be a family of closed subsets of M such that (K1) finite unions and arbitrary non-empty intersections²⁾ of elements of $\mathcal{O}$ belong to $\mathcal{O}$

(K2) $K \subset M$, K compact $\Rightarrow K \in \mathcal{O}$

Then $\mathcal{O}$ is called a K-family in M .

def. 1.2: Let $\mathcal{O}$ and $\mathcal{L}$ be K-families in M .

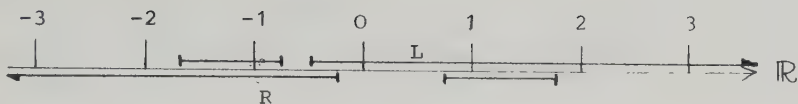
If

(K3) $A \in \mathcal{O}$, $B \in \mathcal{L} \Rightarrow A \cap B$ is compact then $\mathcal{O}$ and $\mathcal{L}$ are called mutually orthogonal.

EXAMPLE: $M = \mathbb{R}$, $L \in \mathcal{L} : \Leftrightarrow L$ is closed and $\inf L > -\infty$,
 $R \in \mathcal{R} : \Leftrightarrow R$ is closed and $\sup R < +\infty$. $\mathcal{L}$ and $\mathcal{R}$ are mutually orthogonal K-families. (Fig. 1.1)

¹⁾ On a compact space the concept of a K-family is trivial: There is only one K-family, the family of compact subsets.

²⁾ An empty intersection of subsets of M is M itself. But in general M does not belong to a K-family. If one would allow arbitrary intersections then a K-family were just a topology defined by its closed sets. This topology would be coarser or equal to the original topology of M .



(Fig. 1.1)

def. 1.3: Let $\mathcal{O}$ and $\mathcal{L}$ be families of closed subsets of M . Then we define the following properties:

(K2') $K \in \mathcal{O} \cap \mathcal{L} \Leftrightarrow K$ is compact

(K3') $A \in \mathcal{O}, B \in \mathcal{L} \Rightarrow A \cap B \in \mathcal{O} \cap \mathcal{L}$

Lemma 1.1: Let $\mathcal{O}$ and $\mathcal{L}$ be families of closed subsets of M . Then

$$(K2) \wedge (K3) \Leftrightarrow (K2') \wedge (K3')$$

Proof: (1) Suppose (K2') and (K3') hold. Let K be compact. Then, by (K2'), $K \in \mathcal{O} \cap \mathcal{L}$, hence $K \in \mathcal{O}$ and $K \in \mathcal{L}$ which is (K2) for $\mathcal{O}$ and $\mathcal{L}$. Let $A \in \mathcal{O}$ and $B \in \mathcal{L}$. From (K3') and (K2') it follows that $A \cap B$ is compact, i.e. (K3).

(2) Suppose (K2) and (K3) hold. In virtue of (K2) every compact K belongs to $\mathcal{O} \cap \mathcal{L}$. The other direction of (K2') is just (K3). Finally let $A \in \mathcal{O}$ and $B \in \mathcal{L}$. From (K3) we know that $A \cap B$ is compact and therefore, by (K2), $A \cap B \in \mathcal{O} \cap \mathcal{L}$. $\square$

Lemma 1.2: The family of compact sets, $\mathcal{K}$, is a K-family and orthogonal to every K-family. $\square$

Corollary 1.3: A K-family $\mathcal{M}$ is orthogonal to itself iff $\mathcal{O} = \mathcal{K}$.

Proof: One part of the proof is Lemma 1.2. For the rest suppose that $\mathcal{O}$ is orthogonal to itself. This means $\mathcal{O} \cap \mathcal{O} = \mathcal{K}$, i.e. $\mathcal{O} = \mathcal{K}$. $\square$

def. 1.4: Let $\mathcal{O}$ be a K-family in M . The dual of $\mathcal{O}$, in symbols $\mathcal{O}^\perp$, is defined by $\mathcal{O}^\perp := \{B \in M \mid B \text{ is closed, } \forall A \in \mathcal{O}: B \cap A \in \mathcal{K}\}$.

Proposition 1.4: $\mathcal{O}^\perp$ is the maximal K-family orthogonal to $\mathcal{O}$.

Proof: (i) $\mathcal{O}^\perp$ is a K-family: Let $(B_i)_{i=1..N} \in \mathcal{O}^\perp$. Then for all $A \in \mathcal{O}$, $A \cap \bigcup_{i=1}^N B_i = \bigcup_{i=1}^N (A \cap B_i)$, which is compact, hence $\bigcup_{i=1}^N B_i \in \mathcal{O}^\perp$. Analogously one shows $\bigcap_{i \in I} B_i \in \mathcal{O}^\perp$ for any non-empty subfamily $(B_i)_{i \in I}$ of $\mathcal{O}^\perp$. This gives (K1). Let $K \in \mathcal{K}$, $A \in \mathcal{O}$. Then $A \cap K \in \mathcal{K}$. Therefore $\mathcal{K} \subset \mathcal{O}^\perp$ which is (K2).

(ii) $\mathcal{O}^\perp$ is orthogonal to $\mathcal{O}$: (K3) holds by definition.

(iii) $\mathcal{O}^\perp$ is maximal: Let $\mathcal{L}$ be a K-family orthogonal to $\mathcal{O}$, $B \in \mathcal{L}$. Then $B \cap A \in \mathcal{K}$ for any $A \in \mathcal{O}$, therefore $B \in \mathcal{O}^\perp$, i.e. $\mathcal{L} \subset \mathcal{O}^\perp$. $\square$

EXAMPLE: Let $\mathcal{C}$ be the family of all closed sets. It is clearly a K-family. We have $\mathcal{C}^\perp = \mathcal{K}$, $\mathcal{K}^\perp = \mathcal{C}$.

Lemma 1.5: $\mathcal{O} \subset \mathcal{L} \Rightarrow \mathcal{L}^\perp \subset \mathcal{O}^\perp$

Proof: $V \in \mathcal{L}^\perp$, i.e. $B \cap V \in \mathcal{K}$ for all $B \in \mathcal{L}$. Since $\mathcal{O} \subset \mathcal{L}$, $B \cap V$ is compact for all $B \in \mathcal{O}$, therefore $V \in \mathcal{O}^\perp$. $\square$

Proposition 1.6: Let $\mathcal{U}$ and $\mathcal{V}$ be families - not necessarily K-families! - of closed sets such that $U \in \mathcal{U}$ and $V \in \mathcal{V}$ imply that $U \cap V \in \mathcal{K}$. Then there exists a K-family $\overline{\mathcal{U}}$ such that $\mathcal{U} \subset \overline{\mathcal{U}}$ and $\mathcal{V} \subset \overline{\mathcal{U}}^\perp$.

Proof: Let $\overline{\mathcal{U}} := \{U \in \mathcal{C} \mid U \cap V \in \mathcal{K} \text{ for all } V \in \mathcal{V}\}$. Obviously $\mathcal{U} \subset \overline{\mathcal{U}}$. $\overline{\mathcal{U}}$ is a K-family: Let $N \in \mathbb{N}$, $(U_i)_{i=1..N} \subset \overline{\mathcal{U}}$, i.e. for all $V \in \mathcal{V}$ and $1 \leq i \leq N$, $U_i \cap V \in \mathcal{K}$. Therefore $(\bigcup_{i=1}^N U_i) \cap V = \bigcup_{i=1}^N (U_i \cap V) \in \mathcal{K}$, whence $\bigcup_{i=1}^N U_i \in \overline{\mathcal{U}}$. Analogously one shows that arbitrary non-empty intersections of elements of $\overline{\mathcal{U}}$ belong to $\overline{\mathcal{U}}$, this gives (K1). Since $\mathcal{V} \subset \mathcal{C}$ by hypothesis and $\mathcal{K} = \mathcal{C}^\perp$ we have $\mathcal{K} \subset \overline{\mathcal{U}}$ which is (K2). Therefore $\overline{\mathcal{U}}$ is a K-family. It is clear from the definition that $\mathcal{V} \subset \overline{\mathcal{U}}^\perp$. $\square$

Remark: In general $\overline{\mathcal{U}} \neq \overline{\mathcal{V}}^\perp$: Let, for example, $\mathcal{U} = \mathcal{V} = \mathcal{K}$. Then $\overline{\mathcal{U}} = \overline{\mathcal{V}} = \mathcal{C}$ but $\overline{\mathcal{V}}^\perp = \mathcal{K}$.

Lemma 1.7: If M is σ -compact (every paracompact manifold is), then every K-family $\mathcal{O}$ contains a sequence $(A_n)_{n \in \mathbb{N}}$ such that

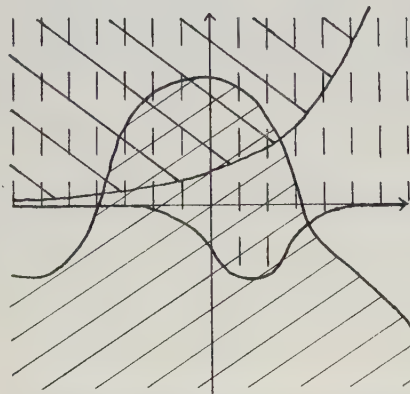
$A_n \subset \text{int}(A_{n+1})$ and $\bigcup_{n=1}^{\infty} A_n = M$.

Proof: $\mathcal{K} \subset \mathcal{O}$ for all K -families $\mathcal{O}$. $\square$

Lemma 1.8: $\mathcal{O} \subset \mathcal{O}^{\perp\perp}$

Proof: $A \cap V$ is compact for all $A \in \mathcal{O}$ and $V \in \mathcal{O}^{\perp}$. $\square$

Remark: $\mathcal{H} = \mathcal{H}^{\perp\perp}$, $e = e^{\perp\perp}$ but in general $\mathcal{O} \neq \mathcal{O}^{\perp\perp}$. Let, for example $M = \mathbb{R}^2$, $A := \{(a_1, a_2) \in \mathbb{R}^2 \mid a_2 \geq 0\}$,



$\mathcal{O} := \{(A \cap L_n) \cup K \mid K \in \mathcal{K}, L \in \mathcal{K} \cup \{M\}\}$.

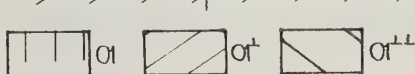
$\mathcal{O}$ is a K -family:

(i) $X \in \mathcal{O} \Rightarrow X$ is closed

(ii) $(K1): \bigcup_{n=1}^N [(A \cap L_n) \cup K_n] =$

$K \cup \bigcup_{n=1}^N (A \cap L_n) = K \cup (A \cap \bigcup_{n=1}^N L_n) =$
 $K \cup (A \cap L),$ where

$K := \bigcup_{n=1}^N K_n, L := \bigcup_{n=1}^N L_n.$



(Fig. 1.2)

$\bigcap_{y \in I} [(A \cap L_y) \cup K_y] =$
 $= \bigcap_{y \in I} [(A \cup K_y) \cap L'] =$
 $= L' \cap \bigcap_{y \in I} (A \cup K_y) = L' \cap (A \cup K) =$
 $= (L' \cap A) \cup K',$ where

$L'_y := L_y \cup K_y, L' := \bigcap_{y \in I} L'_y, K := \bigcap_{y \in I} K_y, K' := L' \cap K, I \neq \emptyset.$

$(K2): L := K \Rightarrow \mathcal{K} \subset \mathcal{O}.$

If $X \in \mathcal{O}$ then every closed subset $Y \subset X$ belongs to $\mathcal{O}^{\perp\perp}$, e.g.

$Y = \{(a_1, a_2) \mid a_2 \geq \exp(a_1)\}$ but $Y \notin \mathcal{O}.$

def. 1.5: A K -family $\mathcal{O}$ is said to be reflexive iff $\mathcal{O}^{\perp\perp} = \mathcal{O}.$

Lemma 1.9: Let $\mathcal{A}$ be an arbitrary family of closed sets. We define

$\mathcal{O}$ by $\mathcal{O} := \{A \in \mathcal{A} \mid A \cap V \text{ is compact for all } V \in \mathcal{A}\}$.

Then $\mathcal{O}$ is a reflexive K-family.

Proof: (1) $\mathcal{O}$ is a K-family: Let $V \in \mathcal{A}$, $(A_n)_{n=1 \dots N} \subset \mathcal{O}$. $(\bigcup_{n=1}^N A_n) \cap V = \bigcup_{n=1}^N (A_n \cap V) \in \mathcal{K}$ since $A_n \cap V \in \mathcal{K}$ for all $1 \leq n \leq N$. Furthermore let $(B_v)_{v \in I} \subset \mathcal{O}$, $I \neq \emptyset$: $(\bigcap_{v \in I} B_v) \cap V = \bigcap_{v \in I} (B_v \cap V) \in \mathcal{K}$ since $B_v \cap V \in \mathcal{K}$ for all $v \in I$. This gives (K1). (K2) is trivial.

(2) $\mathcal{O}^{\perp\perp} = \mathcal{O}$: By definition $\mathcal{A} \subset \mathcal{O}^{\perp}$. Let $B \in \mathcal{O}^{\perp\perp}$. Then $B \cap X \in \mathcal{K}$ for all $X \in \mathcal{O}^{\perp}$, i.e. $B \cap V \in \mathcal{K}$ for all $V \in \mathcal{A}$ which implies by definition $B \in \mathcal{O}$, whence $\mathcal{O}^{\perp\perp} \subset \mathcal{O}$. Lemma 1.8 completes the proof. $\square$

Remark: Lemma 1.9 shows that $\overline{\mathcal{H}}$ as defined in proposition 1.6 is reflexive.

Corollary 1.10: Let $\mathcal{O}$ be a K-family. Then $\mathcal{O}^{\perp}$ and $\mathcal{O}^{\perp\perp}$ are reflexive. $\square$

Lemma 1.11: $\mathcal{O}^{\perp\perp}$ is the minimal reflexive K-family containing $\mathcal{O}$, i.e. $\mathcal{O} \subset \mathcal{L}$ and $\mathcal{L} = \mathcal{L}^{\perp\perp}$ imply $\mathcal{O}^{\perp\perp} \subset \mathcal{L}$.

Proof: $\mathcal{O} \subset \mathcal{L} \Rightarrow \mathcal{L}^{\perp} \subset \mathcal{O}^{\perp} \Rightarrow \mathcal{O}^{\perp\perp} \subset \mathcal{L}^{\perp\perp} = \mathcal{L}$ $\square$

Lemma 1.12: Let $\mathcal{O}$ be reflexive. Then $A \in \mathcal{O}$ and $C \in \mathcal{O}$ imply $A \cap C \in \mathcal{O}$ $\square$

1.2. Locally Convex Topological Vector Spaces - Review of

Definitions and Theorems (Proofs and details may be found in [5, 11, 22, 28] etc.)

Having defined K-families of closed sets we are ready to consider the spaces of tensor fields whose supports are contained in an element of a K-family. These spaces will turn out to be Fréchet spaces. As the properties of these spaces are important for us we first give a short review of definitions and theorems in the theory of locally convex topological vector spaces.

def. 2.1: V is said to be a topological vector space (TVS) iff

(TVS 1) V is a vector space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$

(TVS 2) V is a topological space

(TVS 3) addition and scalar multiplication are continuous.

def. 2.2: Let V be a vector space. A map $p : V \rightarrow \mathbb{R}_+$ is a semi-norm iff

$$(SN\ 1) \quad \forall_{x,y \in V} \quad p(x+y) \leq p(x) + p(y)$$

$$(SN\ 2) \quad \forall_{x \in V} \quad \forall_{\lambda \in \mathbb{K}} \quad p(\lambda x) = |\lambda| p(x)$$

In the following, p shall always (unless otherwise stated) denote a semi-norm.

Proposition 2.1: The following are equivalent

(1) p is continuous

(2) $\{x \in V \mid p(x) < 1\}$ is open

(3) $\{x \in V \mid p(x) < 1\}$ or $\{x \in V \mid p(x) \leq 1\}$
are neighbourhoods of $0 \in V$.

(4) p is continuous at $x = 0$. □

Proposition 2.2: Let V be a TVS. If p and q are semi-norms on V and $q(x) \leq p(x)$ for all $x \in V$ (" $q \leq p$ ") and p is continuous, then q is continuous. □

Proposition 2.3: Let V be a vector space, Λ an (arbitrary) index set and $(p_\alpha)_{\alpha \in \Lambda}$ a family of semi-norms. Then $(p_\alpha)_{\alpha \in \Lambda}$ defines a topology which makes V into a TVS as follows: Let

$$N_{\alpha, r} := \{x \in V \mid p_\alpha(x) < r, \ r > 0, \ \alpha \in \Lambda\} \quad ,$$

then U is a neighbourhood of 0 iff there exist $\alpha_1 \dots \alpha_n$ and $r_1 \dots r_n$ such that $\bigcap_{i=1}^n N_{\alpha_i, r_i} \subset U$ □

def. 2.3: Let V be a vector space. $C \subset V$ is convex iff for all $x, y \in C$ and all $t \in [0, 1]$, $tx + (1-t)y \in C$.

def. 2.4: A TVS is locally convex iff the origin has a fundamental system of convex neighbourhoods.

Proposition 2.4: Let V be a TVS. V is locally convex iff there exists a family of semi-norms $(p_\alpha)_{\alpha \in A}$ defining the topology. $\square$

def. 2.5: A family of semi-norms $\{p_\alpha\}$ on a TVS V is separated iff for all $x \in V \setminus \{0\}$ there exists an α such that $p_\alpha(x) \neq 0$.

Proposition 2.5: Let V be a TVS. V is metrizable iff its topology is defined by a separated countable base of neighbourhoods. $\square$

def. 2.6: A family of semi-norms $(p_\alpha)_{\alpha \in A}$ is directed iff for every $\alpha_1, \dots, \alpha_n \in A$, $n \in \mathbb{N}$, there exists a $\alpha \in A$ such that $p_\alpha \geq \sup \{p_{\alpha_i} \mid i = 1..n\}$.

Proposition 2.6: The topology of a locally convex topological vector space (LCS) may always be defined by a directed family of semi-norms. $\square$

Proposition 2.7: Let V be a LCS, $(p_\alpha)_{\alpha \in A}$ a directed family of semi-norms defining the topology of V and p a semi-norm on V . Then p is continuous if there exists an $\beta \in A$ and a $c > 0$ such that $p \leq cp_\beta$. $\square$

Proposition 2.8: Let V and W be two LCS, $(p_\alpha)_{\alpha \in A}$ a directed family of semi-norms defining the topology of V , $(q_\beta)_{\beta \in B}$ a family of semi-norms defining the topology of W . A linear map $\Lambda: V \rightarrow W$ is continuous iff for every $\beta \in B$ there exists an $\alpha \in A$ and a $c > 0$ such that for all $x \in V$

$$q_\beta(\Lambda[x]) \leq cp_\alpha(x) \quad \square$$

def. 2.7: A Fréchet space is a complete metrizable LCS.

def. 2.8: Let I be an index set. Let V be a vector space¹⁾ such that $V = \bigcup_{\iota \in I} V_\iota$ where $(V_\iota)_{\iota \in I}$ is a family of Hausdorff LCS. We may then define on V the finest locally convex topology such

1) It is important that V is a vector space. This is guaranteed if the (V_ι) possess a "filtrating" property: $\forall \iota, \kappa \in I \quad \exists \lambda \in I \quad V_\iota \subset V_\lambda$ and $V_\kappa \subset V_\lambda$.

that all the inclusion mappings $V_\ell \rightarrow V$ are continuous. A convex subset $U \subset V$ is a neighbourhood of 0 in this topology iff, for every $\ell \in I$, $U \cap V_\ell$ is a neighbourhood of 0 in V_ℓ . When V is equipped with this topology it is called the (interior) inductive limit of the V_ℓ .

Proposition 2.9: Let $V = \bigcup V_\ell$ be endowed with the inductive limit topology (the conventions being as in def. 2.8). A semi-norm p on V is continuous iff its restriction to each V_ℓ is continuous. $\square$

Corollary 2.10: Let W be a LCS, $\Lambda : V \rightarrow W$ a linear map. Then Λ is continuous iff its restriction to each V_ℓ is continuous. $\square$

def. 2.9: Let V be a LCS. V' , the (topological) dual of V is the space of all continuous linear maps $V \rightarrow \mathbb{K}$.

def. 2.10: The weak ("weak-star") topology on V' is the topology defined by the family of semi-norms $(p_x)_{x \in V}$, where $p_x(X) := |X(x)|$ for all $x \in V$, $X \in V'$.

Remark: This is the topology induced on V' by $\mathbb{K}^V \supset V'$, where $\mathbb{K}^V$ is the space of all maps $V \rightarrow \mathbb{K}$ under the topology of pointwise convergence (the product topology).

def. 2.11: Let V be a Fréchet space, W a normed space. Let $\mathbb{L}$ be a set of linear continuous maps $V \rightarrow W$. $\mathbb{L}$ is said to be equicontinuous at $x_0 \in V$ iff, for any $\varepsilon > 0$, there is a $\delta > 0$ such that $d(x_0, x) \leq \delta$ implies for every $\Lambda \in \mathbb{L}$ $\|\Lambda[x_0] - \Lambda[x]\| \leq \varepsilon$

Proposition 2.11: (Theorem of Banach-Steinhaus)

Let V be a Fréchet-space, W a normed space. Let $\mathbb{L}$ be a set of linear continuous maps $V \rightarrow W$. If for all $x \in V$, $\sup_{\Lambda \in \mathbb{L}} \|\Lambda(x)\| < \infty$, then $\mathbb{L}$ is equicontinuous. $\square$

Corollary 2.12: Let $(\Lambda_n)_{n \in \mathbb{N}}$ be a sequence of continuous linear maps $V \rightarrow W$ which converges pointwise in V to a map $\Lambda : V \rightarrow W$. Then Λ is linear and continuous. $\square$

def. 2.12: Let V and W be LCS. $\mathcal{L}(V, W)$ denotes the space of continuous linear mappings $V \rightarrow W$ endowed with the weak¹⁾ topology.

Corollary 2.13: Let Z be a metric space, $A \subset Z$, $A \ni z \mapsto \Lambda_z \in \mathcal{L}(V, W)$. Let $z_0 \in \bar{A}$ and suppose that

$$\lim_{A \ni z \rightarrow z_0} \Lambda_z(x) =: \Lambda(x)$$

exists in W for each $x \in V$. Then $\mathcal{L}(V, W) \ni \Lambda$. □

Corollary 2.14: Let V be a Fréchet space, W a Banach space, $I \subset \mathbb{R}$ an open interval. If a mapping $I \ni t \mapsto \Lambda_t \in \mathcal{L}(V, W)$ is differentiable with respect to the weak¹⁾ topology, then there exists a mapping $I \ni t \mapsto \Lambda'_t \in \mathcal{L}(V, W)$ such that for all $x \in V$

$$\Lambda'_t(x) = D(\Lambda_t(x)) . \quad \square$$

1.3. Test Field Spaces on K-Families

After these preliminaries we are in the position to introduce topological vector spaces of smooth tensor fields satisfying a support condition. Capital gothic letters denote always K-families on M , ∇ a linear connection.

def. 3.1: Let $T \in \mathcal{T}$. $\mathcal{T}_S^r(T)$ is the LCS of $\binom{r}{s}$ tensor fields t on M satisfying the following conditions:

(T 1) $t \in C^\infty(M)$

(T 2) $\text{supp}(t) \subset T$

(T 3) the topology of $\mathcal{T}_S^r(T)$ is defined by the family of seminorms $(p_{D,m})_{(D,m) \in \mathcal{T}^+ \times \mathbb{N}}$ where, e being a Riemannian metric on M ,

$$p_{D,m}(t) := \sup_{\substack{x \in D \\ \alpha \in m}} |\nabla^\alpha t(x)|_e .$$

¹⁾ More precisely: The topology of pointwise convergence. The term "weak" is usually restricted to the case that $F = \mathbb{K}$.

Remark: Whereas the semi-norms depend on e , the topology determined

by $(p_{D,m})$ does not. Equivalent semi-norms are

$$\left\{ \sum_{\alpha \leq m} \sup_{x \in D} |\nabla^\alpha t(x)|_e \right\}$$

and

$$\left\{ \sup_{x \in D} \sum_{\alpha \leq m} |\nabla^\alpha t(x)|_e \right\}$$

Proposition 3.1: The family of semi-norms $(p_{D,m})$ is directed.

Proof: $p_n := p_{D_n, m_n}$, $n = 1 \dots N$. We define $D := \bigcup_{n=1}^N D_n$,
 $m := \max \{m_n \mid n = 1 \dots N\}$. Then obviously $p_{D,m}(t) \geq p_{D_n, m_n}(t)$ for
 all $t \in \mathcal{T}_S^r(T)$, $n = 1 \dots N$. $\square$

Lemma 3.2: A fundamental system of neighbourhoods of 0 is defined
 by

$$N(D, m, \varepsilon; T) := \{t \in \mathcal{T}_S^r(T) \mid p_{D,m}(t) < \varepsilon\} \quad \square$$

Lemma 3.3: $\mathcal{T}_S^r(T)$ has a countable base of neighbourhoods.

Proof: According to Lemma 1.7 there exists a sequence $(D_n)_{n \in \mathbb{N}}$
 such that

$$\bigcup_{n \in \mathbb{N}} D_n = M \text{ and } D_n \subset \text{int } D_{n+1}.$$

$\{N(D_n, m, k^{-1}; T) \mid (n, m, k) \in \mathbb{N}^3\}$ is a countable base. $\square$

Lemma 3.4: The family of semi-norms $(p_{D,m})$ is separated.

Proof: $t \in \mathcal{T}_S^r(T) \setminus \{0\}$, $y \in M$ such that $t(y) \neq 0$. Let C be a
 compact neighbourhood of y (M is locally compact!) then obviously
 $C \in \mathcal{M}^+$ and $p_{C,3}(t) \neq 0$.

Corollary 35: $\mathcal{T}_S^r(T)$ is metrizable.

Proof: Lemmata 3.3, 3.4 and proposition 2.5. $\square$

Lemma 3.6: Let $(t_n)_{n \in \mathbb{N}} \subset \mathcal{T}_S^r(T)$. t_n converges to 0 in $\mathcal{T}_S^r(T)$ iff for all $D \in \mathcal{H}^+$ and for all $\alpha \in \mathbb{N}$, $(\nabla^\alpha t|_D)$ converges uniformly to 0.

Proof: (i) " $\Rightarrow$ " : $t_n \rightarrow 0$ in $\mathcal{T}_S^r(T) \Rightarrow$

$$\forall \varepsilon > 0 \quad \forall m \in \mathbb{N} \quad \forall D \in \mathcal{H}^+ \quad \exists N \in \mathbb{N} \quad \forall n \geq N \quad p_{D,m}(t_n) < \varepsilon, \text{ i.e.}$$

$$\forall \varepsilon > 0 \quad \forall m \in \mathbb{N} \quad \forall D \in \mathcal{H}^+ \quad \exists N \in \mathbb{N} \quad \forall n \geq N \quad \forall \alpha \leq m \quad \forall x \in D \quad |\nabla^\alpha t_n(x)|_e < \varepsilon$$

and, a fortiori,

$$\forall \varepsilon > 0 \quad \forall \alpha \in \mathbb{N} \quad \forall D \in \mathcal{H}^+ \quad \exists N \in \mathbb{N} \quad \forall n \geq N \quad \forall x \in D \quad |\nabla^\alpha t_n(x)|_e < \varepsilon.$$

(ii) " $\Leftarrow$ " : Let $\varepsilon > 0$, $m \in \mathbb{N}$, $D \in \mathcal{H}^+$ be given.

Then, by the condition of uniform convergence,

$$\forall \alpha \in \mathbb{N} \quad \exists N(\alpha) \in \mathbb{N} \quad \forall n \geq N(\alpha) \quad \forall x \in D \quad |\nabla^\alpha t_n(x)|_e < \varepsilon.$$

Let $N := \max_{\alpha \leq m} N(\alpha)$, then

$$\forall n \geq N \quad p_{D,m}(t_n) < \varepsilon. \quad \square$$

Lemma 3.7: $\mathcal{T}_S^r(T)$ is complete.

Proof: Let $(t_n)_{n \in \mathbb{N}} \subset \mathcal{T}_S^r(T)$ such that for all $\alpha \in \mathbb{N}$ and on all $D \in \mathcal{H}^+$, $\nabla^\alpha t_n - \nabla^\alpha t_m \rightarrow 0$ uniformly as $n, m \rightarrow \infty$. As the t_n form a Cauchy sequence of smooth tensor fields (in the uniform convergence norm) they converge on all $D \in \mathcal{H}^+$ uniformly to a smooth tensor field t , and $\nabla^\alpha t_n \rightarrow \nabla^\alpha t$ uniformly on D . It remains to show that $\text{supp}(t) \subset T$: Suppose there exists a $x \in M \setminus T$ such that $t(x) \neq 0$. Since M is locally compact and regular, there is a compact neighbourhood K of x such that $K \cap T = \emptyset$. $K \in \mathcal{H}^+$, $t(x) \neq 0$, but $t_n(x) = 0$ for all $n \in \mathbb{N}$. $\square$

Proposition 3.8: $\mathcal{T}_S^r(T)$ is a Fréchet space.

Proof: proposition 2.4, corollary 3.5, Lemma 3.7 and definition 2.7. $\square$

Lemma 3.9: The family $(\mathcal{T}_S^r(T))_{T \in \mathcal{M}}$ has the following inclusion properties (we denote by $\hookrightarrow$ topological and algebraic inclusion):

$$(1) \mathcal{T}_S^r(T) \hookrightarrow \mathcal{T}_S^r(T') \text{ iff } T \subset T'$$

(2) given T and T' there exists a T'' such that

$$\mathcal{T}_S^r(T) \hookrightarrow \mathcal{T}_S^r(T'')$$

and

$$\mathcal{T}_S^r(T') \hookrightarrow \mathcal{T}_S^r(T'') .$$

Proof: ad(1): Suppose $T \subset T'$. Then $t \in \mathcal{T}_S^r(T)$ implies obviously $t \in \mathcal{T}_S^r(T')$. The continuity is shown with aid of proposition 2.8:

$(p_{D,m})_{(D,m) \in \mathcal{M}^+ \times \mathbb{N}}$ is a directed family of semi-norms defining the topologies of the LCVs $\mathcal{T}_S^r(T)$ and $\mathcal{T}_S^r(T')$ (proposition 3.1).

Denoting the inclusion map by $i_{T',T}$ we have obviously for

all $D \in \mathcal{M}^+$, $m \in \mathbb{N}$ and $t \in \mathcal{T}_S^r(T)$, $p_{D,m}(i_{T',T}[t]) \leq 1 \cdot p_{D,m}(t)$.

Conversely suppose $T \not\subset T'$: Then there is a $x \in T \setminus T'$ and a $t \in \mathcal{T}_S^r(T)$ with $t(x) \neq 0$, i.e. $t \notin \mathcal{T}_S^r(T')$.

ad (2): Choose $T'' := T \cup T'$ and apply (1). $\square$

def. 3.2: $\bigcup_{T \in \mathcal{M}} \mathcal{T}_S^r(T)$ with the inductive limit topology (see def. 2.8) of the $\mathcal{T}_S^r(T)$ is called the space of $(\mathcal{T}_S^r)$ test fields on $\mathcal{M}$ and denoted by $\mathcal{T}_S^r$. We write $\mathcal{D}_S^r$ if $\mathcal{M} = \mathcal{H}$ and $\mathcal{E}_S^r$ if $\mathcal{M} = \mathcal{C}$.

Def. 3.3: A sequence $(t_n)_{n \in \mathbb{N}} \subset \mathcal{T}_s^r$ is said to converge in the sense of $\mathcal{T}$ to 0 ($t_n \xrightarrow{\mathcal{T}} 0$) iff

(C 1) there is a fixed $T \in \mathcal{M}$ such that $t_n \in \mathcal{T}_s^r(T)$ for all $n \in \mathbb{N}$

(C 2) on each $D \in \mathcal{M}^\perp$, $\nabla^\omega t_n$ converges uniformly to zero for all $\omega \in \mathbb{N}$.

Remark: Convergence in the sense of $\mathcal{T}$ implies convergence in $\mathcal{T}$ but is in general stronger. They are the same only if $\mathcal{T}_s^r$ is the countable inductive limit of the $(\mathcal{T}_s^r(T))_{T \in \mathcal{M}}$. In view of Lemma 3.9 this is the case iff there exists a countable subfamily $(T_n)_{n \in \mathbb{N}} \subset \mathcal{M}$ which satisfies the conditions of Lemma 1.7 and has in addition the following property: For every $T \in \mathcal{M}$ there is a $N \in \mathbb{N}$ such that for all $n \geq N$, $T \subset T_n$. A proof of this assertion is analogous to the one given for the space $\mathcal{D}$ in [2], p. 351.

Propositon 3.10: $\mathcal{T}_s^r$ is sequentially¹⁾ complete.

Proof: This is an immediate consequence of def. 3.3 and Lemma 3.7. □

Proposition 3.11: Let $\mathcal{O}, \mathcal{B}$ be K -families. Let $\Lambda: \mathcal{A}_s^r \rightarrow \mathcal{B}_u^t$ be linear.

If for all $A \in \mathcal{O}$ there exists a $B_A \in \mathcal{B}$ such that $\Lambda[\mathcal{A}_s^r(A)] \subset \mathcal{B}_u^t(B_A)$ then Λ is continuous if for all $A \in \mathcal{O}$, $V \in \mathcal{B}^\perp$ and $n \in \mathbb{N}$ there exist a $U \in \mathcal{O}^\perp$, a $m \in \mathbb{N}$ and a $c > 0$ such that for all $a \in \mathcal{A}_s^r(A)$, $p_{V,n}(\Lambda[a]) \leq c p_{U,m}(a)$.

Proof: By definition of the inductive limit (def. 2.8) the inclusion $\mathcal{B}_u^t(B) \rightarrow \mathcal{B}_u^t$ is continuous. Furthermore it follows from

1) sequentially shall always refer to convergence in the sense of $\mathcal{T}$.

corollary 2.10 that Λ is continuous, if $(\bigwedge | \mathcal{A}_S^r(A))$ is continuous for all $A \in \mathcal{O}$. Now the assertion is a consequence of the hypotheses and proposition 2.8. $\square$

Remark: As an example we show the following: The map $\nabla: \mathcal{A}_S^r \rightarrow \mathcal{A}_{S+1}^r$ is continuous.

Proof: Obviously $\text{supp}(\nabla\varphi) \subset \text{supp}(\varphi)$ for all $\varphi \in \mathcal{A}_S^r$. Therefore for all $A \in \mathcal{O}$, $\nabla[\mathcal{A}_S^r(A)] \subset \mathcal{A}_{S+1}^r(A)$

Now, let $A \in \mathcal{O}$, $v \in \mathcal{O}^+$ and $n \in \mathbb{N}$. Then

$$p_{v,n}(\nabla\varphi) \leq p_{v,n+1}(\varphi)$$

for all $\varphi \in \mathcal{A}_S^r(A)$. $\square$

Corollary 3.12: Let $\mathcal{O}, \mathcal{B}$ be K -families, $\mathcal{O} \subset \mathcal{B}$. Then $\mathcal{A}_S^r \subset \mathcal{B}_S^r$.

Proof: $A \in \mathcal{O}$ implies $A \in \mathcal{B}$. Therefore $\mathcal{A}_S^r(A) \subset \mathcal{B}_S^r(A)$ (as a set) for all $A \in \mathcal{O}$, i.e. $\mathcal{A}_S^r \subset \mathcal{B}_S^r$. Proposition 3.11 gives us a sufficient condition for the continuity of the inclusion map. Since $\mathcal{B}^\perp \subset \mathcal{O}^\perp$ the (directed) family of semi-norms defining the topology of $\mathcal{B}_S^r(A)$ is a subfamily of the one defining the topology of $\mathcal{A}_S^r(A)$. Therefore the above inequality may always trivially be satisfied choosing $c = 1$. $\square$

Corollary 3.13: $\mathcal{D}_S^r \subset \mathcal{T}_S^r \subset \mathcal{E}_S^r$

Proof: $\mathcal{H} \subset \mathcal{V} \subset \mathcal{C}$ $\square$

Proposition 3.14: $\mathcal{D}_S^r$ is dense in $\mathcal{T}_S^r$.

Proof: M is σ -compact. Therefore there exists a sequence

$(K_n)_{n \in \mathbb{N}} \subset \mathcal{K}$ such that $M = \bigcup K_n$, $K_n \subset \text{int } K_{n+1}$. Then for every $T \in \mathcal{M}$ we have $Q_n := T \cap K_n \in \mathcal{K}$. Let $\varphi_n : M \rightarrow [0, 1]$ be a smooth function such that $(\varphi_n / K_n) = 1$ and $\text{supp } (\varphi_n) \subset K_{n+1}$. Now, let $t \in \mathcal{T}_S^r(T)$ and let $t_n := \varphi_n t$. Obviously t_n is smooth and $\text{supp}(t_n) \subset Q_{n+1}$, therefore $t_n \in \mathcal{D}_S^r(Q_{n+1}) \subset \mathcal{D}_S^r$. Furthermore $t_n \xrightarrow{\mathcal{T}} t : (C 1)$ is trivial since for all $n \in \mathbb{N}$, $Q_n = T \cap K_n \subset T \in \mathcal{M}$. Now, let $D \in \mathcal{M}^\perp$. $D \cap T$ is compact, therefore there exists a $N \in \mathbb{N}$ such that for all $n \geq N$, $(D \cap T) \subset \text{int } Q_n$. Hence, for $n \geq N$ and $x \in D$

$$\nabla^\alpha t_n(x) = \nabla^\alpha (\varphi_n t)(x) = \nabla^\alpha t(x)$$

since $(\varphi_n|_{D \cap T}) = 1$ for $n \geq N$. $\square$

Corollary 3.15: $\mathcal{M} \subset \mathcal{L}$ implies $\mathcal{A}_S^r$ is dense in $\mathcal{B}_S^r$.

Proof: $\mathcal{D}_S^r \subset \mathcal{A}_S^r \subset \mathcal{B}_S^r$ and $\mathcal{D}_S^r$ is dense in $\mathcal{B}_S^r$. $\square$

1.4. Distributions Based on K-Families

In the preceding section we defined TVSS of tensor fields satisfying a support condition and we derived their most important properties. We now will look at their topological duals. We shall find that the topological dual of the space of $\binom{r}{s}$ test fields on $\mathcal{M}$ consists of the "generalized $\binom{s}{r}$ tensor fields" (tensor distributions) whose support is an element of $\mathcal{M}^\perp$. This is precisely the result we need in order to apply the whole well-known calculus of distributions to these spaces. This means that the mathematical apparatus which we want to apply to our problem, will have been developed by the end of this section.

Def. 4.1: $(\mathcal{T}_s^r)'$ denotes the topological dual of $\mathcal{T}_s^r$, i.e. the space of all continuous linear functionals on $\mathcal{T}_s^r$. $(\mathcal{T}_s^r)'$ is given the weak topology, i.e. $L_n \rightarrow 0$ iff $\lim_{n \rightarrow \infty} \langle L_n, t \rangle = 0$ for all $t \in \mathcal{T}_s^r$.

Remark: If one applies this definition to $\mathcal{D} = \mathcal{D}_0^0$ one obtains $\mathcal{D}'$, the space of (ordinary) distributions.

Proposition 4.1: Let $L : \mathcal{T}_s^r \rightarrow \mathbb{K}$ be linear. $L \in (\mathcal{T}_s^r)'$ iff $\lim_{n \rightarrow \infty} \langle L, t_n \rangle = 0$ whenever $t_n \xrightarrow{\mathcal{T}_s^r} 0$.

Proof: According to corollary 2.10 L is continuous iff its restriction to each $\mathcal{T}_s^r(T)$, $T \in \mathcal{V}$, is continuous. Since $\mathcal{T}_s^r(T)$ is metrizable $(L|_{\mathcal{T}_s^r(T)})$ is continuous iff $\langle L, t_n \rangle \rightarrow 0$ for all $t_n \rightarrow 0$ in the topology of $\mathcal{T}_s^r(T)$. Definition 3.3 completes the proof. $\square$

Proposition 4.2: (Semi-norm estimate) Let $L : \mathcal{T}_s^r \rightarrow \mathbb{K}$ be linear. $L \in (\mathcal{T}_s^r)'$ iff for all $T \in \mathcal{V}$ there exist a $D \in \mathcal{V}^+$, a $c > 0$ and a $N \in \mathbb{N}$ such that for all $t \in \mathcal{T}_s^r(T)$

$$|\langle L, t \rangle| \leq c \cdot p_{D,N}(t)$$

Proof: Corollary 2.10 and proposition 2.8. $\square$

Proposition 4.3: Let $\mathcal{O} \subset \mathcal{B}$. Then $(\mathcal{B}_s^r)' \subset (\mathcal{A}_s^r)'$.

Proof: Let $L \in (\mathcal{B}_s^r)'$, $a \in \mathcal{A}_s^r$, $i_{\mathcal{B},\mathcal{A}} : \mathcal{A}_s^r \rightarrow \mathcal{B}_s^r$ the inclusion map.

Obviously

$$\langle L, a \rangle := \langle L, i_{\mathcal{B},\mathcal{A}}[a] \rangle$$

Since $i_{\mathcal{B},\mathcal{A}}$ and $L : \mathcal{B}_s^r \rightarrow \mathbb{K}$ are linear and continuous (corollary 3.12), $L \circ i_{\mathcal{B},\mathcal{A}} : \mathcal{A}_s^r \rightarrow \mathbb{K}$ is linear and continuous, i.e. $(\mathcal{B}_s^r)' \subset (\mathcal{A}_s^r)'$. It remains to show the continuity of the inclusion map. An open ball with centre 0 in $(\mathcal{A}_s^r)'$ is given by

$$\{L \in (\mathcal{A}_S^r)' : |\langle L, a \rangle| < \varepsilon \text{ for } a \in \mathcal{Z}\},$$

$\mathcal{Z}$ being a finite subset of $\mathcal{A}_S^r$. If one chooses a $L \in (\mathcal{B}_S^r)'$ one sees that this is also an open ball in $(\mathcal{B}_S^r)'$ since $\mathcal{A}_S^r \subset \mathcal{B}_S^r$. $\square$

Proposition 4.4: $L \in (\mathcal{T}_S^r)'$ is equivalent with $L \in (\mathcal{D}_S^r)'$ and $\text{supp } (L) \in \mathcal{M}^+$.

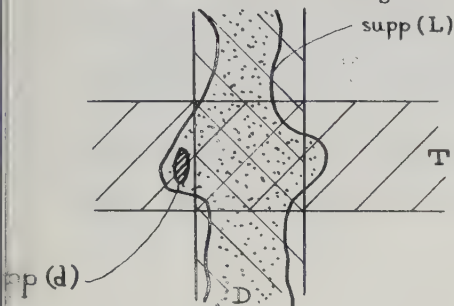
Proof: (i) suppose that $L \in (\mathcal{D}_S^r)'$ and $\text{supp } (L) \in \mathcal{M}^+$. L induces a linear functional on $\mathcal{T}_S^r$ in the following way: Let $t \in \mathcal{T}_S^r$. Then there is a $T \in \mathcal{M}$ such that $t \in \mathcal{T}_S^r(T)$. By definition $K := T \cap \text{supp } (L)$ is compact. Take a smooth function $\varphi_t : M \rightarrow [0, 1]$ such that $\varphi_t = 1$ in a neighbourhood of K and $\text{supp } (\varphi_t) \in \mathcal{K}$. Then $\varphi_t t \in \mathcal{D}_S^r$ and $\varphi_t t = t$ in a neighbourhood of K . We define $\langle L, t \rangle := \langle L, \varphi_t t \rangle$. It is clear that this does not depend on the particular choice of φ_t . Therefore L is a well defined linear functional on $\mathcal{T}_S^r$. Furthermore let $t_n \xrightarrow{\mathcal{T}} 0$. (C 1) implies that one may choose one φ_t for all t_n such that $\langle L, t_n \rangle = \langle L, \varphi_t t_n \rangle$. From (C 2) it follows that $\nabla^{\alpha} t_n \xrightarrow{\mathcal{D}} 0$ uniformly on every compact set. We deduce from the Leibnitz rule and the fact that $\text{supp } (\varphi_t)$ is compact that on every compact set and for all $\alpha \in \mathbb{N}$, $\nabla^{\alpha} (\varphi_t t_n) \rightarrow 0$ uniformly, i.e. $\varphi_t t_n \xrightarrow{\mathcal{D}} 0$. Therefore $\langle L, t_n \rangle = \langle L, \varphi_t t_n \rangle \rightarrow 0$, i.e. $L \in (\mathcal{T}_S^r)'$.

(ii) $L \in (\mathcal{T}_S^r)'$. From corollary 3.13 and proposition 4.3 it follows that $L \in (\mathcal{D}_S^r)'$. It remains to show that for all $T \in \mathcal{M}$, $\text{supp } (L) \cap T$ is compact: Let $T \in \mathcal{M}$. Then by the semi-norm estimate of proposition 4.2 there exist a $D \in \mathcal{M}^+$, a $c > 0$ and a $n \in \mathbb{N}$ such that for all $t \in \mathcal{T}_S^r(T)$

$$|\langle L, t \rangle| \leq c \sup_{\substack{x \in D \\ \alpha \leq n}} |\nabla^{\alpha} t(x)|_e$$

Suppose that $\text{supp } (L) \cap T \not\subset D \cap T$.

Then there is a $d \in \mathcal{D}_S^r$ such that $\text{supp}(d) \subset T$, $\text{supp}(d) \cap D = \emptyset$ and



(Fig.4.1)

$\langle L, d \rangle \neq 0$. Obviously $d \in \mathcal{T}_S^r(T)$, therefore, by the above estimate

$$|\langle L, d \rangle| \leq c \cdot 0$$

whence

$$\langle L, d \rangle = 0 !$$

We deduce from this contradiction that $\text{supp}(L) \cap T \subset D \cap T$, which is compact. T and $\text{supp}(L)$ are closed. Therefore $\text{supp}(L) \cap T$ is compact. $\square$

Remark: The elements of $(\mathcal{T}_S^r)'$ may be considered as generalized $(\frac{S}{r})$ tensor fields with support in an element of $\mathcal{M}^+$: Let f be a locally integrable $(\frac{S}{r})$ tensor field. Then f induces a distribution in $(\mathcal{T}_S^r)'$ by

$$\mathcal{T}_S^r \ni t \mapsto \int f \cdot t \eta =: \langle f, t \rangle$$

Proof: Obviously $t \mapsto \int f \cdot t \eta$ is linear. Let $T \in \mathcal{M}$ be given. Then, for all $t \in \mathcal{T}_S^r(T)$

$$|\int f \cdot t \eta| \leq \int_T |f| e \eta \cdot \sup_{x \in \text{supp}(f)} |t(x)|_e$$

Proposition 4.2 completes the proof. $\square$

In the following we shall not distinguish a locally integrable tensor field from the distribution that it induces.

Remark: We deduce from proposition 4.4 that the support conditions on the distribution and the test field spaces are complementary and determine each other in the following sense if $\mathcal{H}$ is reflexive: Given a reflexive K-family $\mathcal{D}$, the subspace of $(\mathcal{D}_S^r)'$ consisting of all $L \in (\mathcal{D}_S^r)'$ such that $\text{supp}(L) \in \mathcal{D}$, is equal to $(\mathcal{T}_S^r)'$ where $\mathcal{H} = \mathcal{D}^\perp$. This follows directly from proposition 4.4 and the reflexivity of $\mathcal{D}$, i.e. $\mathcal{H}^\perp = \mathcal{D}^{\perp\perp} = \mathcal{D}$. On the other side, starting with $\mathcal{H}, \mathcal{T}_S^r$, passing to $\mathcal{H}^\perp, (\mathcal{T}_S^r)'$ and applying the above procedure one obtains again $\mathcal{H}, \mathcal{T}_S^r$ iff $\mathcal{H} = \mathcal{H}^{\perp\perp}$. For this reason we shall consider in the following only reflexive K-families. This is no real restriction of generality: Suppose, we begin with a non-reflexive K-family $\mathcal{H}$ and want to construct the test field and the distribution spaces over $\mathcal{H}$. In this case we pass first to $\mathcal{H}^{\perp\perp}$ which is reflexive (corollary 1.10) and construct then $\mathcal{T}_S^r$ and $(\mathcal{T}_S^r)'$. The distribution space obtained in that way is the same as the one we had obtained if we started directly from $\mathcal{H}$. This follows from proposition 4.4 and the fact that $\mathcal{H}^{\perp\perp\perp} = \mathcal{H}^\perp$ (corollary 1.10).

Remark: Proposition 4.4 allows us to apply the whole well-known theory of distributions also to the elements of any $(\mathcal{T}_S^r)'$, for example the notions of covariant derivative, order etc. Therefore, the theory of distributions based on K-families has now been completely developed.

Def. 4.2: A distribution $L \in (\mathcal{T}_S^r)'$ is of finite order iff there is a $N \in \mathbb{N}$ such that for all $T \in \mathcal{H}$ there exist a $D \in \mathcal{H}^\perp$ and a $c > 0$ such that for all $t \in \mathcal{T}_S^r(T)$

$$|\langle L, t \rangle|_{\mathcal{C}} p_{D,N}(t)$$

The smallest such N is called the order of the distribution.

EXAMPLE: Let $n \in \mathbb{N}$ and let f be a locally integrable $\binom{s}{r}$ tensor field with $\text{supp}(f) \in \mathcal{M}^1$. f induces a distribution in $(\mathcal{T}_s^{r+n})'$ of finite order $\leq n$ by

$$\mathcal{T}_s^{r+n} \ni t \mapsto (-1)^n \int f \cdot (\nabla_s^n t) \eta$$

This distribution is the generalized (distributional) derivative of f and will also be denoted by $\nabla^n f$ (confer the first remark after proposition 4.4).

1.5. Some Remarks on K-Family Based Distributions on Submanifolds which are the Closure of Open Subsets of the Manifold M

The construction we are going to perform will be well defined only in a neighbourhood of a timelike wordtube due to the finiteness of the radius of injectivity of the exponential map. In order to deal with this difficulty we inquire about the relation inbetween the distribution spaces belonging to a submanifold which is the closure of an open subset of M and the distribution spaces belonging to M itself.

Def. 5.1: A subset N of the differentiable manifold M is said to be a submanifold of M iff the inclusion map $i : N \rightarrow M$ is a differentiable embedding¹⁾.

¹⁾ This implies that N is given the subspace topology of M .

In the following N is always considered to be a submanifold of M . Furthermore we shall assume that N is the closure of an open subset of M .

def. 5.2: Let $\mathcal{O}$ be a K -family in M . We define its restriction to N by

$$\mathcal{O}_N := \{A \cap N \mid A \in \mathcal{O}\}$$

Proposition 5.1: $\mathcal{O}_N$ is a K -family in N .

Proof: $B \in \mathcal{O}_N$ implies that $B = A \cap N$ for some $A \in \mathcal{O}$. Since A is closed in M , B is closed in N . Let $(B_i)_{i=1 \dots n} \subset \mathcal{O}_N$. Then there are $(A_i)_{i=1 \dots n} \subset \mathcal{O}$ such that $B_i = A_i \cap N$ for all $i = 1 \dots n$.

We deduce from (K1) that $A := \bigcup_{i=1}^n A_i \in \mathcal{O}$. Therefore

$$\bigcup_{i=1}^n B_i = \bigcup_{i=1}^n (A_i \cap N) = \left(\bigcup_{i=1}^n A_i \right) \cap N = A \cap N \in \mathcal{O}_N.$$

Analogously one shows that $\bigcap_{i \in I} B_i \in \mathcal{O}_N$ if $I \neq \emptyset$ and $(B_i)_{i \in I} \subset \mathcal{O}_N$.

This gives (K1). Now, let $K \subset N$ be compact in N . Since N is closed in M , K is compact in M . Therefore $K \in \mathcal{O}$. Hence $K = K \cap N \in \mathcal{O}_N$ which is (K2). $\square$

Proposition 5.2: $(\mathcal{O}_N)^\perp = (\mathcal{O}^\perp)_N$

Proof: (i) Let $W \in (\mathcal{O}_N)^\perp$, i.e. $W \cap A \cap N$ is compact for all $A \in \mathcal{O}$. But $W \subset N$. Hence $W \cap A$ is compact for all $A \in \mathcal{O}$, i.e. $W \in \mathcal{O}^\perp$ and therefore $W = W \cap N \in (\mathcal{O}^\perp)_N$.

(ii) Conversely, let $W \in (\mathcal{O}^\perp)_N$, i.e. $W = V \cap N$ for some V such that $V \cap A$ is compact for all $A \in \mathcal{O}$. One confirms immediately that $W \cap A \cap N = V \cap A \cap N$ is compact for all $A \in \mathcal{O}$. $\square$

def. 5.3: $\mathcal{O}_N^\perp := (\mathcal{O}^\perp)_N = (\mathcal{O}_N)^\perp$

Corollary 5.3: $\mathcal{O}_N$ is reflexive whenever $\mathcal{O}$ is reflexive.

Proof: Using proposition 5.2 we find $(\mathcal{O}_N)^{\perp\perp} = (\mathcal{O}^\perp)_N^\perp = (\mathcal{O}^{\perp\perp})_N = \mathcal{O}_N$. $\square$

After these remarks concerning the relation of K-families on the manifold to K-families on the submanifold we consider next the corresponding test field spaces. According to the second remark after proposition 4.4 all K-families considered are reflexive.

def. 5.4: Let $\mathcal{M}$ be a reflexive K-family in M , $\mathcal{T}_S^r$ the $(\mathcal{T}_S^r)$ test field space on $\mathcal{M}$. We define ${}_N\mathcal{T}_S^r$ to be the corresponding test field space on $\mathcal{M}_N$.

Lemma 5.4: The family of semi-norms defining the topology of ${}_N\mathcal{T}_S^r$ $(T \cap N)$, $T \in \mathcal{M}$, is the family of semi-norms $\{p_{D,m} \mid D \in \mathcal{M}^\perp, m \in \mathbb{N}\}$ restricted to ${}_N\mathcal{T}_S^r(T \cap N)$.

Proof: This is a simple consequence of proposition 5.2. $\square$

Proposition 5.5: A distribution $L \in ({}_N\mathcal{T}_S^r)'$ with $\text{supp}(L) \in \mathcal{M}^\perp$ and $\text{supp}(L) \subset \text{int } N$ may be extended to a distribution in $(\mathcal{T}_S^r)'$. In order to make this extension unique we define it to be 0 on $M \setminus N$. In this case we shall not distinguish L from its extension.

Proof: According to proposition 4.3 $L \in ({}_N\mathcal{D}_S^r)'$. Let $\varphi : M \rightarrow [0,1]$ be a smooth function such that $\varphi = 1$ in a neighbourhood of $\text{supp}(L)$ and $\text{supp}(\varphi) \subset \text{int } N$. Then for all $t \in \mathcal{D}_S^r$, $\varphi t \in {}_N\mathcal{D}_S^r$. Using Lemma 5.4, one shows easily that $\mathcal{D}_S^r \ni t \mapsto \langle L, \varphi t \rangle \in \mathbb{K}$ is a well defined continuous linear functional independent on the particular choice of φ (confer the proof of proposition 4.4), i.e. $L \in (\mathcal{D}_S^r)'$. Proposition 4.4 completes the proof. $\square$

2. THE ENERGY-MOMENTUM DISTRIBUTION, THE TEST FIELD SPACES AND THEIR DECOMPOSITION

2.1 Geometry, the Basic K-Family and the Basic Function Space

We consider a system of finitely many extended bodies. Matter shall be described by a divergence-free distribution $T \in (\mathcal{D}'_{[2]}^0)$. Since we take into account only spatially compact systems there exists a timelike worldtube W such that $\text{supp}(T) \subset W$. For technical reasons we suppose that there exist a timelike worldline $l \equiv z(s)$ contained in W and a timelike future directed (t.f.d.) unit vector field $u(s)$ along l . We define for $z \in M$, $x \in M_z$,

$$\Sigma(z, x) := \left\{ x \in M \mid x^k \sigma_k(z, x) = 0 \right\}$$

the geodesic hypersurface orthogonal to X . Furthermore let

$$\Sigma(s) := \Sigma(z(s), u(s)).$$

We suppose that there exists a submanifold $N \subset M$ and that l, u and N satisfy the following conditions

- (i) N is the closure of an open subset of M
- (ii) $W \subset \text{int } N$
- (iii) two spacelike geodesics emerging from a point $x \in N$ do not reintersect within N
- (iv) for all $t, r \in \mathbb{R}$, $r \leq t$,

$$\bigcup_{r \leq s \leq t} \Sigma(s) \cap W \text{ is compact}$$
- (v) for $r \neq s$, $\Sigma(r) \cap \Sigma(s) \cap N = \emptyset$, i.e. it is possible to define a time function τ on N by $\tau(x) = t \iff x \in \Sigma(t)$.

In addition we require $\Sigma(s) \cap W$ to be star-shaped with respect to $z(s)$ for all $s \in \mathbb{R}$, i.e. for each $x \in \Sigma(s) \cap W$ the geodesic segment joining x and $z(s)$ is entirely contained in $\Sigma(s) \cap W$. This is

always the case when W is spacelike geodesically convex.

Now we want to define a distribution space based on a K -family, which incorporates the above support condition and whose corresponding test field space is closed under certain operations which are necessary to construct the skeleton: $\mathcal{Q}_{[2]}^{\circ}$ is not closed under these operations, since they involve integration along spacelike geodesics such that the compactness of the supports is destroyed. In fact this was the motivation to develop the concept of K -family based distribution spaces.

In order to define the distribution space we construct a K -family and the corresponding test field space. Now the problem arises what K -family shall we choose? The most obvious attempt would be to define one general K -family for all kinds of systems of extended bodies in the following way: Let

$$\mathcal{M} := \{W \in \mathcal{C} \mid W \text{ satisfies the above conditions}\}$$

and

$$\mathcal{Q} := \{G \in \mathcal{C} \mid G \cap W \in \mathcal{K} \text{ for all } W \in \mathcal{M}\}$$

In virtue of Lemma 1.1.9 $\mathcal{Q}$ is a reflexive K -family. $\mathcal{G}_s^r$ denotes the $\binom{r}{s}$ test field space over $\mathcal{Q}$. Then

$$\{T \in (\mathcal{G}_{[2]}^{\circ})' : \langle T, \omega \rangle = 0 \quad \forall \omega \in \mathcal{G}_1^{\circ}\}$$

is the appropriate energy momentum distribution space. This definition sounds very natural, but it is not difficult to see that we have not got what we wanted: A generic element of $\mathcal{Q}$ is the sandwich between two global Cauchy surfaces. If the space-time does not admit global Cauchy surfaces then we find $\mathcal{Q} = \mathcal{K}$

and therefore $\mathcal{G}_s^r = \mathcal{D}_s^r$, what we wanted to avoid from the beginning. Furthermore we loose the support condition since $(\mathcal{G}_{[2]}^0)' = (\mathcal{D}_{[2]}^0)'$. Additionally, even when global Cauchy surfaces exist, the hypersurfaces $\Sigma(s)$ are not necessarily global cauchy surfaces, which creates again problems with the above mentioned operations on the test field space.

For these reasons we shall adopt the following attitude: As we only study one system at a time (i.e. we keep the number of bodies finite), we are concerned with one world tube W at a time. So, with every particular worldtube W containing a system of spatially compact extended bodies we associate its own test field space. This creates no problems since those test function spaces play a rôle only for the theoretical considerations leading to the construction of the skeleton $\hat{T}$. We proceed in the following way:

def. 1.1: Let W be a timelike worldtube containing the system under consideration and satisfying the above regularity conditions. Then we define

$$\mathcal{F}_W := \{F \in \mathcal{E} \mid F \cap W \in \mathcal{K}\}$$

to be the basic K -family of the system. ${}^W\mathcal{F}_s^r$ denotes the space of the $(\overset{r}{s})$ test fields on $\mathcal{F}_W$.

Remark: We consider from now on one arbitrary but fixed system. Therefore we may suppress the index W without any risk of confusion.

Lemma 1.1: $T \in (\mathcal{D}_{[2]}^0)'$, $\text{supp}(T) \subset W$ and $\nabla \cdot T = 0$ imply $T \in (\mathcal{F}_{[2]}^0)'$ and $\langle T, \mathcal{I}\omega \rangle = 0$ for all $\omega \in \mathcal{F}_1^0$.

Proof: Trivial consequence of proposition 1.4.4. $\square$

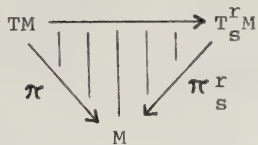
As we are concerned only with distributions with support in $W \subset \text{int } N$ we may without lack of generality suppose $N = M$: This follows from the above hypotheses and proposition 1.5.5. Furthermore we choose $\mathbb{K}$ to be the real field $\mathbb{R}$.

2.2 Auxiliary Test Field Spaces

The skeleton will be a distribution on the tangent space. Therefore we need also test field spaces on the tangent space. They are in normal neighbourhoods equivalent to two point tensor fields. In this section we are going to describe the properties of these spaces.

def. 2.1: Let $\pi_s^r : T_s^r M \rightarrow M$ be the natural projection, $\pi = \pi_0^1$.

A map $E : TM \rightarrow T_s^r M$ is said to be a $\binom{r}{s}$ tensor field over π , or



(Fig. 2.1.)

$E \in (T_s^r, T)M$, iff $\pi_s^r \circ E = \pi$, i.e.

the diagram shown in fig. 2.1.

commutes. For $z \in M$ and $X \in M_z$ we

write $E_{\substack{k_1, \dots, k_r \\ l_1, \dots, l_s}}(z, X)$, where the

indices refer to a basis in M_z .

If there is no risk of confusion we shall simply write (T_s^r, T) instead of $(T_s^r, T)M$.

Proposition 2.1: There is a natural differential operator ∇_* :

$(T_s^r, T) \rightarrow (T_{s+1}^r, T)$, the vertical covariant derivative, defined as follows: M_z is isomorphic to $\mathbb{R}^4$; therefore one is given a natural flat connection on M_z whose covariant derivative operator we denote by ∇_* . $\square$

def. 2.2: A cross section e of $T_S^r M \otimes T_Q^p M$ (i.e. a map $e : M \times M \rightarrow T_S^r M \otimes T_Q^p M$ such that $(\pi_S^r \otimes \pi_Q^p) \cdot e = \text{id}_{M \times M}$) is said to be a two-point tensor field on M (bi-tensor field), we write $e \in T_{S,Q}^{r,p} M$. If the arguments x and z are restricted to subsets $X \subset M$ and $Z \subset M$, we write $T_{S,Q}^{r,p}(X \times Z)$. In this notation holds obviously $T_{S,Q}^{r,p} M = T_{S,Q}^{r,p}(M \times M)$.

Definitions and properties of the most important bi-tensor fields are found in appendix A of this section.

Proposition 2.2: The exponential map provides a one-one correspondence between (T_S^r, T) and $T_{O,S}^{O,r} M$ on a neighbourhood of the diagonal set in $M \times M$.

Proof: Let first be given a bi-scalar $\beta(z, x)$: There corresponds a scalar field B over π by

$$B(z, X) := \beta(z, \text{Exp}_Z X)$$

Applying the vertical derivative operator ∇_* yields

$$\nabla_{*k} B(z, X) = H_k^a \nabla_a \beta(z, \text{Exp}_Z X)$$

Generalized we have the correspondence

$$A = \text{Exp}_Z^* \alpha$$

or

$$A_{1 \dots 1}^{k_1 \dots k_r} (z, X) = (-1)^r \sigma_{a_1 \dots a_r}^{k_1 \dots k_r} H_{1_1}^{b_1} \dots H_{1_s}^{b_s} \alpha_{b_1 \dots b_s}^{a_1 \dots a_r}$$

where $A \in (T_S^r, T)$ and $\alpha \in T_{O,S}^{O,r} M$. □

Remark: An other possibility would, of course, be the following:

$$A = \text{Exp}_{Z*}^{-1} \alpha .$$

Various subspaces of (T_S^r, T) and $T_{q,s}^{p,r}M$ having analogous properties as the spaces introduced in 1.3 may be defined using (possibly different) support properties with respect to both arguments. As those spaces play only an auxiliary rôle and as their properties in this application are quite straightforward, we are not going to formalize them.

Appendix A of Section 2.2: The Most Important Bi-Tensor Fields.

For accounts on the theory of bi-tensors see [4] and [26]. We follow closely the notation of Dixon as reported for example in the appendix A1 of [7]. We just summarize the most important definitions and properties: For a bi-tensor function of the point pair (x, z) we must distinguish indices at x and indices at z . We shall use letters from the beginning of the alphabet for indices at x and letters from the middle of the alphabet at z .

def. 2.A1: If two points $x, z \in M$ can be joined by a unique geodesic $y(u)$, with $y(u_1) = z$ and $y(u_2) = x$ we define the world function bi-scalar $\sigma(z, x)$ by

$$\sigma(z, x) := \frac{1}{2}(u_2 - u_1) \int_{u_1}^{u_2} g_{ab}(y(u)) \frac{dy^a}{du} \frac{dy^b}{du} du$$

Its derivatives play an important rôle. We denote them simply by appropriate suffixes, thus

$$\frac{\partial}{\partial x^a} \sigma(z, x) = \sigma_a(z, x)$$

$$\frac{\partial}{\partial z^k} \sigma(z, x) = \sigma_k(z, x)$$

and so on. In index-free notation we write for the covariant derivative with respect to x , ∇_x etc.: This means that the a -th component of $\nabla_x \sigma$ is σ_a .

def. 2.A2: Let x, z and $y(u)$ be defined as above. Let, furthermore, ξ be a Jacobi field along $y(u)$. Then $\xi(y(u))$ satisfies the equation of geodesic deviation along $y(u)$:

$$D_u^2 \xi^a + R^a_{\dot{y}b} \xi^b = 0 \quad (2.A0)$$

where $\dot{y} = D_u y \cdot \xi$, as a solution of (2.A0) is determined by the values of ξ and $D_u \xi$ at z . It follows from the theory of linear differential equations that there must be propagators K and H such that

$$\xi^a(y(u)) = K^a_k(z, y(u)) \xi^k + u H^a_k(z, y(u)) D \xi^k$$

K and H are called first and second Jacobi propagators.

Proposition 2.A1: The Jacobi propagators are expressible in terms of the world function:

$$H_a^k = -\sigma^{-1} \sigma_a^k \quad (2.A1)$$

$$K_a^k = H_a^1 \sigma_1^k \quad (2.A2)$$

Furthermore

$$H = \text{Exp} \star$$

Proof: See [6] and [21] .

def. 2.A3: Let A be a two-point tensor field. We define $\langle A \rangle$, the coincidence limit of A at z , by

$$\langle A \rangle := \lim_{x \rightarrow z} A(z, x)$$

Lemma 2.A2: (J.L. Synge, [26]) Let A be a two-point tensor field.

Then

$$\nabla \langle \nabla_x^n \nabla_z^m A \rangle = \langle \nabla_x^{n+1} \nabla_z^m A \rangle + \langle \nabla_x^n \nabla_z^{m+1} A \rangle$$

Proof: Let $x(u, t)$ be a 1-parameter family of geodesics, t being the family parameter and u being an affine parameter.

Let $x(0,t) =: z(t)$, $X := \partial_t x(u,t)$, $\dot{z} = \partial_t z$. Then

$$D_t \nabla_x^n \nabla_z^m A(z(t), x(u,t)) = X \cdot \nabla_x^{n+1} \nabla_z^m A + \dot{z} \cdot \nabla_z^{m+1} \nabla_x^n A \quad (2.A3)$$

Obviously

$$\lim_{u \rightarrow 0} X = \dot{z}$$

Therefore

$$\dot{z} \cdot \nabla \langle \nabla_x^n \nabla_z^m A \rangle = \dot{z} \cdot \langle \nabla_x^{n+1} \nabla_z^m A \rangle + \dot{z} \cdot \langle \nabla_z^{m+1} \nabla_x^n A \rangle \quad (2.A4)$$

(2.A3) holds for all one-parameter families of geodesics. This means that (2.A4) holds for arbitrary vectors z whence the assertion follows. $\square$

Proposition 2.A3 [26]: The coincidence limits of the world function and its derivatives up to degree 4 are

$$\langle \sigma \rangle = 0 \quad (2.A5)$$

$$\langle \nabla_x \sigma \rangle = \langle \nabla_z \sigma \rangle = 0 \quad (2.A6)$$

$$\langle \nabla_x^2 \sigma \rangle = \langle \nabla_z^2 \sigma \rangle = -\langle \nabla_{xz} \sigma \rangle = \delta \quad (2.A7)$$

$$\langle \nabla_x^n \nabla_z^{3-n} \sigma \rangle = 0 \quad \text{for } 0 \leq n \leq 3 \quad (2.A8)$$

$$\langle \sigma_{abcd} \rangle = \frac{1}{3} (R_{adbc} + R_{acbd}) \quad (2.A9)$$

$$\langle \sigma_{kabc} \rangle = \frac{1}{3} (R_{kabc} + R_{kbac}) \quad (2.A10)$$

$$\langle \sigma_{klab} \rangle = -\frac{1}{3} (R_{kab l} + R_{kba l}) \quad (2.A11)$$

$$\langle \sigma_{klma} \rangle = \frac{1}{3} (R_{aklm} + R_{alkm}) \quad (2.A12)$$

$$\langle \sigma_{klmn} \rangle = \frac{1}{3} (R_{knlm} + R_{kmln}) \quad (2.A13)$$

Proof: (2.A5) and (2.A6) are trivial from the definition. Differentiation of the identity

$$\sigma^a \sigma_a = \sigma^k \sigma_k = 2\sigma$$

yields

$$\sigma^a \sigma_{ab} = \sigma_b, \quad \sigma^k \sigma_{kl} = \sigma_l \quad (2.A14)$$

$$\sigma^a \sigma_{ak} = \sigma_k, \quad \sigma^k \sigma_{ka} = \sigma_a$$

Whence (2.A7) follows. We differentiate (2.A14) again and get

$$\sigma^e_c \sigma_{eb} + \sigma^e \sigma_{ebc} = \sigma_{bc}$$

$$\sigma^e_{ca} \sigma_{eb} + \sigma^e_c \sigma_{eba} + \sigma^e_a \sigma_{ebc} + \sigma^e \sigma_{ebca} = \sigma_{bca} \quad (2.A15)$$

It follows that

$$\langle \sigma_{b(ac)} \rangle = 0 \quad (2.A16)$$

But

$$\sigma_{a[bc]} = \frac{1}{2} R_{bca}^d \sigma_d \quad (2.A17)$$

whence

$$\langle \sigma_{b[ac]} \rangle = 0 \quad (2.A18)$$

(2.A16) and (2.A18) together give (2.A8) for $n = 3$. Application of Lemma 2.A2 and of equations (2.A7) yields (2.A8) for $0 \leq n \leq 2$. We obtain by a further differentiation of (2.A15)

$$\sigma^e_{cad} \sigma_{eb} + \sigma^e_{ca} \sigma_{ebd} + \sigma^e_{cd} \sigma_{eba} + \sigma^e_c \sigma_{ebad} + \sigma^e_{ad} \sigma_{ebc} +$$

$$\sigma^e_a \sigma_{ebcd} + \sigma^e_d \sigma_{ebca} + \sigma^e \sigma_{ebcad} = \sigma_{bcd}$$

Hence

$$\langle \sigma_{bacd} \rangle + \langle \sigma_{bcad} \rangle + \langle \sigma_{bdca} \rangle = 0 \quad (2.A19)$$

Observing that

$$\sigma_{abcd} - \sigma_{abdc} = R_{cdb}^e \sigma_{ae} + R_{cda}^e \sigma_{eb}$$

we find

$$\langle \sigma_{abcd} \rangle = \langle \sigma_{abdc} \rangle$$

Differentiating (2.A17) we obtain

$$\sigma_{abcd} - \sigma_{acbd} = \nabla_d R_{bca}^e \sigma_e + R_{bca}^e \sigma_{ed}$$

$$\langle \sigma_{abcd} \rangle - \langle \sigma_{acbd} \rangle = R_{bcad} \quad (2.A20)$$

$$\langle \sigma_{abdc} \rangle - \langle \sigma_{adbc} \rangle = R_{bdac} \quad (2.A21)$$

(2.A21) has been obtained from (2.A20) interchanging c and d.

Adding these two equations and using (2.A19) we find (2.A9). With Lemma 2.A2 and equation (2.A8) we get (2.A10) ... (2.A13). $\square$

Corollary 2.A4: The coincidence limits of the Jacobi propagators and their derivatives up to order 2 are

$$\langle K \rangle = \delta \quad (2.A22)$$

$$\langle \nabla_x^K \rangle = \langle \nabla_z^K \rangle = 0 \quad (2.A23)$$

$$\langle \nabla_{cb} K^a_k \rangle = R^a_{bkc} \quad (2.A24)$$

$$\langle H \rangle = \delta \quad (2.A25)$$

$$\langle \nabla_x^H \rangle = \langle \nabla_z^H \rangle = 0 \quad (2.A26)$$

$$\langle \nabla_{cb} H^a_k \rangle = \frac{2}{3} R^a_{(kb)c} \quad (2.A27)$$

Furthermore

$$\langle \nabla_{*mn} H_k^a \rangle = \frac{2}{3} R^a_{(mn)k} \quad (2.A28)$$

Proof: Differentiating

$$H_k^a \sigma_b^k = -\delta_b^a \quad (2.A29)$$

we obtain

$$\nabla_b H_k^a = H_{|1}^a \sigma_{eb}^1 H_k^e \quad (2.A30)$$

$$\nabla_{cb} H_k^a = H_{|1}^a \sigma_{ebc}^1 H_k^e + (\nabla_c H_{|1}^a) \sigma_{eb}^1 H_k^e + H_{|1}^a \sigma_{eb}^1 \nabla_c H_k^e \quad (2.A31)$$

From (2.A29) and (2.A7) we find (2.A25), from (2.A30) and (2.A8) we find (2.A26) and from (2.A31), (2.A10) and (2.A26) we find (2.A27). Analogously we obtain from successive differentiation of (2.A2) the equations (2.A22), (2.A23) and (2.A24). Finally we get

$$\begin{aligned} \langle \nabla_{*mn} H_k^a \rangle &= \langle H_{(m}{}^c \nabla_c (H_{|n)}^b \nabla_b H_k^a) \rangle = \\ &= \delta_{(m}^e \delta_{n)}^b \langle \nabla_{eb} H_k^a \rangle = \frac{2}{3} R^a_{(mn)k} \quad \square \end{aligned}$$

def. 2.A4: Let $G := \text{Exp}_z^* g$, i.e.

$$G_{kl}(z, x) = H_k^a(z, x) H_l^b(z, x) g_{ab}(x)$$

where $x = \text{Exp}_z X$. In a neighbourhood of $X = 0$, G is a metric on M_z . We denote the Christoffel symbols of the corresponding Levi-Civita connexion by G_{kl}^m .

Lemma 2.A5:

$$G_{kl}^m = H_k^a H_l^b \sigma_{ab}^m \quad (2.A32)$$

Proof: See [10], equation (128). $\square$

Lemma 2.A6:

$$\langle G_{kl}^m \rangle = 0 \quad (2.A33)$$

$$\langle \nabla_{*m} G_{kl}^n \rangle = \frac{2}{3} R^n_{(kl)m} \quad (2.A34)$$

Proof: (2.A33) follows from Lemma 2.A5 and (2.A8). Differentiation of (2.A32) yields

$$\begin{aligned} \langle \nabla_{*m} G_{kl}^n \rangle &= \delta_m^c \langle \nabla_c (-\sigma_b^n H_k^a H_l^b) \rangle = \\ &= \delta_m^c \delta_b^n \delta_k^a \langle \nabla_{ca} H_l^b \rangle = \frac{2}{3} R^n_{(kl)m}. \end{aligned} \quad \square$$

2.3 Composable Families of Distributions

From now on we consider $W, l, u, \Sigma(s), \tau, \mathcal{F}, \mathcal{F}_S^T$ etc. to be given as described in section 2.1.

We want that momentum P and angular momentum S are smooth tensor fields along l and that the skeleton is a real-valued distribution. We will see that for an arbitrary energy-momentum distribution T this is not necessarily satisfied. There is, however, a possibility to characterize a certain class of distributions the so-called composable families - such that the above requirements are satisfied. Therefore we introduce and develop the theory of composable families of distributions in the following. In order to do that we have to give sense to "the restriction of a distribution to a hypersurface". Throughout this section we suppose $\{X(s), s \in \mathbb{R}\}$ to be a spacelike foliation of M such that

- (i) for all compact intervals $I \subset \mathbb{R}$, $[\bigcup_{s \in I} X(s)] \cap W$ is compact.
- (ii) The "time-function" $\chi: M \rightarrow \mathbb{R}$ defined by $\chi(x) = s \iff x \in X(s)$ is of class C^∞ .

In the following sections - unless otherwise stated - we shall always take $\{X(s)\}$ to be the foliation $\{\Sigma(s)\}$ determined by a smooth curve l and a vector field u along it, as described in section 2.1.

In order to define a "restriction of a distribution to the hypersurfaces $X(s)$ " we multiply the test fields by functions which are constant on the hypersurfaces and look at the resulting functional:

def. 3.1: Let $\varphi \in \mathcal{E}_S^r$ and $f \in \mathcal{E}(\mathbb{R})$. We define $\varphi \perp f := \varphi f \cdot \chi$

Lemma 3.1: Let $F \in \mathcal{F}$. Then $\varphi \in \mathcal{F}_S^r(F)$ implies $\varphi \perp f \in \mathcal{F}_S^r(F)$ for all $f \in \mathcal{E}(\mathbb{R})$. $\square$

Lemma 3.2: Let $\varphi \in \mathcal{E}_S^r$ and $f \in \mathcal{D}(\mathbb{R})$. Then $\varphi \perp f \in \mathcal{F}_S^r(\chi^{-1}(\text{supp}(f)))$. $\square$

Proposition 3.3: $\perp : \mathcal{F}_S^r \times \mathcal{E}(\mathbb{R}) \rightarrow \mathcal{F}_S^r$ (resp. $\mathcal{E}_S^r \times \mathcal{D}(\mathbb{R}) \rightarrow \mathcal{F}_S^r$) is bilinear and partially continuous.

Proof: The linearity is trivial. Now let $f \in \mathcal{E}(\mathbb{R})$, $F \in \mathcal{F}$, $v \in \mathcal{F}^\perp$ and $n \in \mathbb{N}$ be given. Then for all $\varphi \in \mathcal{F}_S^r(F)$

$$\begin{aligned} p_{V,n}(\varphi \perp f) &= \sup_{\substack{x \in V \\ \alpha \leq n}} \left| \nabla^\alpha (\varphi(x) f(\chi(x))) \right|_e \leq \\ &\sup_{\substack{x \in V \cap F \\ \alpha \leq n}} \left\{ \left| \nabla^\alpha \varphi(x) \right|_e \left| f(\chi(x)) \right| + \dots + \right. \\ &\quad \left. + \left| \varphi(x) \right|_e \left| \nabla^\alpha f(\chi(x)) \right|_e \right\} \leq C p_{V,n}(\varphi) \end{aligned}$$

Proposition 1.3.11 shows that $\varphi \mapsto \varphi \perp f$ is continuous for fixed f . With similar estimates one proves the other assertions. $\square$

Lemma 3.4: Let $\varphi \in \mathcal{E}_S^r$; $\psi \in \mathcal{E}_P^q$; $f, h \in \mathcal{E}(\mathbb{R})$. Then

$$(\varphi \perp f) \otimes (\psi \perp h) = (\varphi \otimes \psi) \perp (fh)$$

$$(\varphi \perp f) \perp h = \varphi \perp (fh) \quad \square$$

Lemma 3.5: Let $\varphi \in \mathcal{E}_S^r$, $f \in \mathcal{E}(\mathbb{R})$. Then

$$\nabla (f \circ \chi) = \partial_\chi \perp_{DF}$$

$$\nabla (\varphi \perp f) = \nabla \varphi \perp f + \partial_\chi \otimes \varphi \perp_{DF} f \quad \square$$

Proposition 3.6: Let $L \in (\mathcal{F}_s^r)'$, $\varphi \in \mathcal{E}_s^r$ (resp $\mathcal{F}_s^r$). Then there exists a distribution $\int_{\mathcal{X}} L \varphi \in \mathcal{D}'(\mathbb{R})$ (resp. $\mathcal{E}'(\mathbb{R})$) defined by

$$\langle \int_{\mathcal{X}} L \varphi \mid f \rangle := \langle L, \varphi \perp f \rangle$$

Proof: The map $\mathcal{D}(\mathbb{R}) \ni f \mapsto \langle \int_{\mathcal{X}} L \varphi \mid f \rangle$ is the composition of $f \mapsto \varphi \perp f$ and $\psi \mapsto \langle L, \psi \rangle$. Both are linear and continuous (proposition 3.3), hence the composition is linear and continuous, i.e. it is a distribution. The rest of the assertion is clear from the support properties of $\varphi \perp f$. $\square$

Remark: $\varphi \mapsto \int_{\mathcal{X}} L \varphi$ is in some sense the "restriction of L to $\{\mathcal{X}(s)\}$ ": This may be seen as follows: Let the distribution L be represented by a differentiable tensor field, i.e.

$$\langle L, \varphi \rangle = \int L_{a_1 \dots a_r}^{b_1 \dots b_s}(x) \varphi_{b_1 \dots b_s}^{a_1 \dots a_r}(x) \eta(x)$$

Suppose that $f \in \mathcal{E}(\mathbb{R})$. Then

$$\langle \int_{\mathcal{X}} L \varphi \mid f \rangle = \int L_{a_1 \dots a_r}^{b_1 \dots b_s}(x) \varphi_{b_1 \dots b_s}^{a_1 \dots a_r}(x) f(\chi(x)) \eta(x) \quad (3.1)$$

Furthermore holds

$$\eta = w^a dx_a \wedge ds$$

where dx_a is the vector element of volume on $\mathcal{X}(s)$ and w is a vector field such that

$$w(\chi) = 1$$

Now we may write (3.1) as

$$\begin{aligned} \langle \int_{\mathcal{X}} L \varphi \mid f \rangle &= \\ &= \int ds f(s) \int_{\mathcal{X}(s)} L_{a_1 \dots a_r}^{b_1 \dots b_s}(x) \varphi_{b_1 \dots b_s}^{a_1 \dots a_r}(x) w^c dx_c \end{aligned}$$

whence

$$\int_X L \varphi = \int_X L_{a_1 \dots a_r}^{b_1 \dots b_s} \varphi_{b_1 \dots b_s}^{a_1 \dots a_r} w^c dx_c$$

This shows also that our notation is quite appropriate.

Proposition 3.7: $\int_X : (\mathcal{F}_S^r)' \times \mathcal{F}_S^r \rightarrow \mathcal{E}'(\mathbb{R})$ (resp. $(\mathcal{F}_S^r)' \times \mathcal{E}_S^r \rightarrow \mathcal{D}'(\mathbb{R})$) is bilinear and partially continuous.

Proof: The linearity is trivial. Let $L \in (\mathcal{F}_S^r)'$ be fixed. The map $\mathcal{F}_S^r \ni \varphi \mapsto \int_X L \varphi \in \mathcal{E}'(\mathbb{R})$ is continuous if its restriction to each $\mathcal{F}_S^r(F)$, $F \in \mathcal{F}$, is continuous (corollary 1.2.10). In virtue of proposition 1.2.8 we have to show that for all $f \in \mathcal{E}(\mathbb{R})$ there exist a $V \in \mathcal{F}^\perp$, a $n \in \mathbb{N}$ and a $C > 0$ such that for all $\varphi \in \mathcal{F}_S^r(F)$

$$| \langle \int_X L \varphi \mid f \rangle | \leq C p_{V,n}(\varphi) \quad (3.2)$$

But (3.2) is a direct consequence of propositions 1.4.2 and 3.3.

Now let $\varphi \in \mathcal{F}_S^r$ be fixed. Let $f \in \mathcal{E}(\mathbb{R})$, $\varepsilon > 0$. We define

$$S_{\varepsilon, f} := \{ Q \in \mathcal{E}'(\mathbb{R}) \mid |\langle Q \mid f \rangle| < \varepsilon \}$$

Then, by definition of the weak topology, $\mathcal{O}$, defined by

$$\mathcal{O} := \{ S_{\varepsilon, f} \mid \varepsilon > 0, f \in \mathcal{E}(\mathbb{R}) \}$$

is a subbase of the topology of $\mathcal{E}'(\mathbb{R})$. Therefore $\Lambda : L \mapsto \int_X L \varphi$ is continuous if for all $f \in \mathcal{E}(\mathbb{R})$ and $\varepsilon > 0$, $\Lambda^{-1}(S_{\varepsilon, f})$ is open in $(\mathcal{F}_S^r)'$. But

$$\Lambda^{-1}(S_{\varepsilon, f}) = \{ L \in (\mathcal{F}_S^r)' \mid |\langle L, \varphi \mid f \rangle| < \varepsilon \}$$

which is by definition an open subset of $(\mathcal{F}_S^r)'$. □

def. 3.2: For $Q \in \mathcal{E}'(\mathbb{R})$ we write

$$\int Q := \langle Q \mid 1 \rangle$$

$\underline{1}$ being the function $x \mapsto 1$ for all $x \in \mathbb{R}$.

In the following we show that the "restriction of a distribution to $\{\chi(s)\}$ " may be manipulated in the same way as if it were simply the integral of a C^∞ tensor field over a hypersurface.

Lemma 3.8: Let $L \in (\mathcal{F}_s^r)'$, $\varphi \in \mathcal{F}_s^r$. Then

$$\langle L, \varphi \rangle = \iint_{\chi} L \varphi.$$

Remark: This generalizes

$$\int L(x) \cdot \varphi(x) \eta(x) = \int ds \int_{\chi(s)} L(x) \cdot \varphi(x) w.dX$$

Proof:

$$\begin{aligned} \iint_{\chi} L \varphi &= \langle \int_{\chi} L \varphi \mid \underline{1} \rangle = \\ &= \langle L, \varphi \mid \underline{1} \rangle = \langle L, \varphi \rangle \quad \square \end{aligned}$$

Lemma 3.9: $L \in (\mathcal{F}_s^r)'$, $\varphi \in \mathcal{F}_s^r$. Then

$$\int^D \int_{\chi} L \varphi = 0$$

Remark: This generalizes

$$\int ds \frac{d}{ds} \int_{\chi(s)} L \cdot \varphi w.dX = 0$$

Proof:

$$\begin{aligned} \int^D \int_{\chi} L \varphi &= \langle \int_{\chi} L \varphi \mid \underline{1} \rangle = \\ &= - \langle \int_{\chi} L \varphi \mid {}^D \underline{1} \rangle = 0. \quad \square \end{aligned}$$

Lemma 3.10: Let $L \in (\mathcal{F}_s^r)'$, $\varphi \in \mathcal{E}_s^r$, $h \in \mathcal{E}(\mathbb{R})$.

Then

$$\int_X L(\varphi \perp h) = \left(\int_X L\varphi \right) h$$

Remark: This generalizes

$$\int_{X(s)} L\varphi \cdot h \circ \chi \quad w \cdot dX = h(s) \int_{X(s)} L\varphi \quad w \cdot dX$$

Proof: Let $f \in \mathcal{D}(\mathbb{R})$. Then

$$\begin{aligned} \left\langle \int_X L(\varphi \perp h) \mid f \right\rangle &= \left\langle L, (\varphi \perp h) \perp f \right\rangle = \\ &= \left\langle L, \varphi \perp (hf) \right\rangle = \left\langle \int_X L\varphi \mid hf \right\rangle = \\ &= \left\langle \left(\int_X L\varphi \right) h \mid f \right\rangle. \quad \square \end{aligned}$$

Lemma 3.11: Let $L \in (\mathcal{F}_{s+1}^r)'$, $\varphi \in \mathcal{E}_s^r$. Then

$$D \int_X L \cdot \partial_X \varphi = \int_X \nabla L \varphi + \int_X L \nabla \varphi.$$

Remark: This generalizes

$$\begin{aligned} \frac{d}{ds} \int_{X(s)} \varphi_{b_1 \dots b_s}^{a_1 \dots a_r} L_{a_1 \dots a_r}^{b_1 \dots b_{s+1}} dx_{b_{s+1}} &= \\ &= \int_{X(s)} \varphi_{b_1 \dots b_s}^{a_1 \dots a_r} \nabla_{b_{s+1}} L_{a_1 \dots a_r}^{b_1 \dots b_{s+1}} w^c dx_c + \\ &+ \int_{X(s)} \nabla_{b_{s+1}} \varphi_{b_1 \dots b_s}^{a_1 \dots a_r} L_{a_1 \dots a_r}^{b_1 \dots b_{s+1}} w^c dx_c \end{aligned}$$

Proof: Let $f \in \mathcal{D}(\mathbb{R})$. Then

$$\begin{aligned} \left\langle D \int_X L \cdot \partial_X \varphi \mid f \right\rangle &= \\ &= - \left\langle \int_X L \cdot \partial_X \varphi \mid Df \right\rangle = \\ &= - \left\langle L \cdot \partial_X \varphi \perp Df \right\rangle = \end{aligned}$$

$$\begin{aligned}
&= - \langle L, \partial \chi \otimes \varphi \perp df \rangle = \\
&= - \langle L, \nabla(\varphi \perp f) - \nabla \varphi \perp f \rangle = \\
&= \langle \nabla \cdot L, \varphi \perp f \rangle + \langle L, \nabla \varphi \perp f \rangle = \\
&= \langle \int_X \nabla \cdot L \varphi \mid f \rangle + \langle \int_X L \varphi \mid f \rangle. \quad \square
\end{aligned}$$

def. 3.3: A distribution $L \in (\mathcal{T}_s^r)'$ is said to be composed of a composable family of distributions on $\{\chi(s)\}$ or restrictable to $\{\chi(s)\}$ iff

(CF 1) for all $\varphi \in \mathcal{E}_s^r$ the distribution $\int_X L \varphi$ is equivalent to a smooth function $\mathbb{R} \ni s \mapsto \int_{\chi(s)} L \varphi$, i.e. for $f \in \mathcal{D}(\mathbb{R})$

$$\langle \int_X L \varphi \mid f \rangle = \int ds f(s) \int_{\chi(s)} L \varphi \quad (3.3)$$

(CF 2) for all $\varphi_n \xrightarrow{\mathcal{E}} 0$ and all $s \in \mathbb{R}$ holds

$$\lim_{n \rightarrow \infty} \int_{\chi(s)} L \varphi_n = 0 \quad (3.4)$$

Remark: (CF 2) follows almost everywhere from (CF 1): Let $f \in \mathcal{D}(\mathbb{R})$,

$\varphi_n \xrightarrow{\mathcal{E}} 0$. Then $\varphi_n \perp f \xrightarrow{\mathcal{F}} 0$ in virtue of proposition 3.3 and thus

$$\lim_{n \rightarrow \infty} \langle \int_X L \varphi_n \mid f \rangle = 0$$

From (3.3) we get

$$\lim_{n \rightarrow \infty} \int ds f(s) \int_{\chi(s)} L \varphi_n = 0$$

for all $f \in \mathcal{D}(\mathbb{R})$. Hence

$$\int_{\chi(s)} L \varphi_n \rightarrow 0 \text{ a.e.}$$

Remark: The concept of composable families of distributions was introduced by R. Christescu in [3] for families of distributions in $\mathcal{D}'(\mathbb{R})$ depending on a real parameter.

EXAMPLES:

- (1) C^∞ fields are restrictable to all $\{\chi(s) : s \in \mathbb{R}\}$
- (2) finite linear combinations of derivatives of the 3-dim. δ -distribution with support along a smooth timelike curve $\gamma(s)$ contained in W , i.e.

$$\int_{\chi(s)} L \varphi = \sum_{\alpha \leq N} c_\alpha(s) \nabla^\alpha \varphi(\gamma(s))$$

are restrictable to all $\{\chi(s) : s \in \mathbb{R}\}$.

("2^N-pole particle")

Proposition 3.12: If L is restrictable to $\{\chi(s)\}$ then for all $s \in \mathbb{R}$, $\varphi \mapsto \int_{\chi(s)} L \varphi$ is a distribution with compact support contained in $\chi(s) \cap \text{supp}(L)$.

Proof: In view of (CF 1) $\varphi \mapsto \int_{\chi(s)} L \varphi$ is a well defined linear map $\mathcal{E}_s^r \rightarrow \mathbb{R}$. From (3.4) follows the continuity (proposition 1.4.1). Finally, let $\varphi \in \mathcal{E}_s^r$ vanish in an open neighbourhood K of $\chi(s) \cap \text{supp}(L)$. Since $\chi(s) \cap \text{supp}(L)$ is compact we can find a $\varepsilon > 0$ such that $\bigcup_{- \varepsilon + s \leq t \leq s + \varepsilon} \chi(t) \cap \text{supp}(L) \subset K$. Now let $f \in \mathcal{D}(\mathbb{R})$ with $\text{supp}(f) \subset [s - \frac{1}{2}\varepsilon, s + \frac{1}{2}\varepsilon]$.

It follows that

$$\text{supp}(\varphi \perp f) = \bigcup_{- \varepsilon \leq s - t \leq \varepsilon} \chi(t) \cap K$$

whence it is immediately clear that

$$\text{supp}(L) \cap \text{supp}(\varphi \perp f) = \emptyset$$

i.e. for all $f \in \mathcal{D}(\mathbb{R})$ with $\text{supp}(f) \subset [s - \frac{\epsilon}{2}, s + \frac{\epsilon}{2}]$ we find

$$\int dt f(t) \int_{\chi(t)} L \Phi = 0$$

or, for all $\Phi \in \mathcal{E}_s^T$ vanishing in a neighbourhood of $\chi(s) \cap \text{supp}(L)$

$$\int_{\chi(s)} L \Phi = 0. \quad \square$$

Remark: In general $\text{supp}(\Phi \mapsto \int_{\chi(s)} L \Phi) \neq \chi(s) \cap \text{supp}(L)$:

Let, for example, L be represented by $\psi \chi$, where ψ is some smooth scalar function with $\text{supp}(\psi) = W$. Since $\text{supp}(\chi) = M$ we have $\text{supp}(L) = W$. But

$$\int_{\chi(0)} L \Phi = \int_{\chi(0)} \psi(x) \chi(x) \Phi(x) w^a dx_a = 0$$

i.e. $\text{supp}(\Phi \mapsto \int_{\chi(0)} L \Phi) = \emptyset \neq W \cap \chi(0)$.

Proposition 3.13: If L is restrictable to $\{\chi(s)\}$ then

$\Phi \mapsto \left(\frac{d}{ds}\right)^n \int_{\chi(s)} L \Phi$ is a distribution with compact support contained in $\chi(s) \cap \text{supp}(L)$ for all $s \in \mathbb{R}$ and all $n \in \mathbb{N}$.

Proof: This follows immediately from proposition 3.12 and corollary 1.2.14, which, obviously, may be generalized to inductive limits of Fréchet spaces. $\square$

Finally we need a definition similar to def. 3.3 for the dual spaces of our auxiliary fields (the property of the skeleton of a restrictable energy momentum distribution).

def. 3.4: Let $\lambda: s \mapsto \zeta(s)$ be a smooth curve and let n be a smooth unit vector field along λ . Let furthermore $H_{\zeta(s)} := \text{Exp}_{\zeta(s)}^{-1} \Sigma(\zeta(s), n(s))$. A family of distributions $\{S_x \in \mathcal{D}'(T_x M) \mid x \in \lambda(\mathbb{R})\}$ is said to be smoothly composable with respect to (λ, n) iff

$$(SC\ 1) \quad \text{supp } (S_{\mathcal{Y}(s)}) \subset H_{\mathcal{Y}(s)}$$

$$(SC\ 2) \quad s \mapsto \langle S_{\mathcal{Y}(s)}, \Phi \rangle \quad \text{is smooth for all} \\ \Phi \in \mathcal{D}(\text{TM}).$$

This definition is easily extended to any type of fields in $(T_s^r, T)M$ based on any K -family in TM .

Proposition 3.14: Let $\{S_x\}$ be a family of distributions smoothly composable with respect to (λ, n) . Then $\{\Phi \mapsto (\frac{d}{ds})^n \langle S_{\mathcal{Y}(s)}, \Phi \rangle \mid s \in \mathbb{R}\}$ is smoothly composable with respect to (λ, u) . $\square$

2.4 The Decomposition of $(T_{[2]}^0, T)M$

In the following two sections we attack the main problem: How to extract the free information contained in a distribution in $(\mathcal{F}_{[2]}^0)'$ obeying the local law of motion $\nabla \cdot T = 0$? This is obviously equivalent to the problem of the free information contained in a distribution $T \in (\mathcal{F}_{[2]}^0)'$ such that $\mathcal{G}_{[2]}^0 := \{\$ \omega \mid \omega \in \mathcal{F}_1^0\} \subset \ker(T)$. Therefore we look for a splitting of $\mathcal{F}_{[2]}^0$ into the direct sum of $\mathcal{G}_{[2]}^0$ and some complementary subspace, $\mathcal{F}_{[2]}^0 = \mathcal{G}_{[2]}^0 \oplus \mathcal{T}_{[2]}^0$. This will be the first step towards the construction of the skeleton. We also need an analogous splitting for $(T_{[2]}^0, T)$ since the skeleton $\hat{T}$ will be a distribution on this space obeying $\nabla_* \cdot \hat{T} = 0$. M_x is, in addition, flat such that certain complications are not present in the construction of the splitting. Therefore we first treat the case of the decomposition of $(T_{[2]}^0, T)$. The results we will obtain in the

following two sections are in their original form due to W.G. Dixon [9]. The proofs given here are partially different. Furthermore we introduce projection operators, consider continuity and support properties and give explicit expressions for all involved quantities.

We use Taylor's formula with integral rest as base of our splitting theorem. This formula also enables us to find explicit expressions for our projection operators. Due to the importance of this formula for the following we quote it here. A proof may be found for example in [1], p. 77.

Proposition 4.1 (Taylor's formula): Let U be an open subset of a Banach space, F a Banach space and $f : U \rightarrow F$ a map of class C^{N+1} . If the segment $[a, a+h]$ is contained in U , then

$$\begin{aligned} f(a+h) &= \sum_{n=0}^N (n!)^{-1} f^{(n)}(a) \cdot (h)^n + \\ &+ \int_0^1 (N!)^{-1} (1-t)^N f^{(N+1)}(a+th) \cdot (h)^{N+1} dt \quad \square \end{aligned} \quad (4.1)$$

In the following considerations we shall look at $(T_{[2]}^0, T)$ over one arbitrary but fixed point $z \in M$. For typographic convenience we suppress the argument z in expressions like $E(z, X)$ if there is no risk of confusion. We use d_* and S_* to denote exterior and symmetrized derivatives with respect to the vertical covariant derivative.

Proposition 4.2: Let $\Psi \in (T_{[2]}^0, T)$ such that for all $X \in M_z$, $X \cdot \Psi(X) = 0$. Furthermore let

$$H_{klmn}(X) := 4 \int_0^1 u(1-u) \nabla_{*[k[l} \Psi_{m]n]}(uX) du$$

where antisymmetrization is taken over (k,m) and (l,n) separately.

Then

$$\Psi_{kl}(X) = X^m X^n H_{kmnl}(X) \quad (4.2)$$

Proof: Differentiation of $X \cdot \Psi(X) = 0$ yields

$$\Psi_{mk} + X^n \nabla_{*m} \Psi_{nk} = 0 \quad (4.3)$$

Antisymmetrization gives

$$X^n \nabla_{*[m} \Psi_{k]n} = 0$$

Differentiating again and multiplying with X^m we find

$$X^m \nabla_{*[m} \Psi_{k]l} = X^m X^n \nabla_{*l[k} \Psi_{m]n} \quad (4.4)$$

In the case $N = 1$ Taylor's formula (4.1) applied to a function $f: \mathbb{R} \rightarrow \otimes_2^0 M_Z$ reads

$$f(1) = f(0) + f'(0) + \int_0^1 (1-u) f''(u) du \quad (4.5)$$

We put $f(u) := u \Psi(uX)$. Then

$$f(0) = 0, \quad f(1) = \Psi(X) \quad (4.6)$$

$$f'(u) = \Psi(uX) + u \nabla_{*X} \Psi(uX) \quad (4.7)$$

Substitution of X by uX in (4.3) gives, when inserted in (4.7),

$$f'_{kl}(u) = 2uX^m \nabla_{*[m} \Psi_{k]l}(uX) \quad (4.8)$$

whence

$$f'(0) = 0 \quad (4.9)$$

Differentiating (4.8) we find

$$f''_{kl}(u) = 2 X^m \nabla_{*[m} \Psi_{k]l} + 2u X^m X^n \nabla_{*n[m} \Psi_{k]l} \quad (4.10)$$

Finally we substitute $X \rightarrow uX$ in (4.4) and insert the result into (4.10):

$$f''_{kl}(u) = u X^m X^n \left\{ \nabla_{*mn} \Psi_{kl} - \nabla_{*nk} \Psi_{ml} - \right. \\ \left. - \nabla_{*lm} \Psi_{kn} + \nabla_{*lk} \Psi_{mn} \right\}$$

or

$$f''_{kl}(u) = 4u X^m X^n \nabla_{*[k[l} \Psi_{m]n]}(uX) \quad (4.11)$$

where antisymmetrization is taken over (k,m) and (l,n) separately.

If we put (4.11) into (4.5) using (4.6) and (4.7), we find (4.2) to be valid. $\square$

Proposition 4.3: Let $\Psi \in (T^0_{[2]}, T)$. Then the following three are equivalent:

- (i) $X^k \Psi_{kl}(X) = 0$ for all $X \in M_Z$
- (ii) there exists a $H \in (T^0_{[2,2]}, T)$ such that $\Psi_{kl}(X) = X^m X^n H_{kmnl}(X)$
- (iii) $X^k X^l \nabla_{*\{k} \Psi_{m]l}\}(X) = 0$ (4.12)

and

$$\Psi_{kl}(0) = 0 \quad (4.13)$$

Proof: (i) $\Rightarrow$ (ii) : Proposition 4.2.

(ii) $\Rightarrow$ (i) : Obvious from the symmetry properties of H.

(i) $\wedge$ (ii) $\Rightarrow$ (iii) : (4.13) follows directly from (ii).

Application of ∇_{*m} to (i) and transvection with X^m and X^1 respectively gives

$$X^k X^m \nabla_{*m} \Psi_{kl}(X) = 0$$

$$X^k X^1 \nabla_{*m} \Psi_{kl}(X) = 0$$

which imply (iii).

(iii) $\Rightarrow$ (i) : Multiplication of (4.12) by X^m yields

$$X^k X^1 X^m \nabla_{*k} \Psi_{lm}(X) = 0$$

Along a straight line $u \mapsto X = uZ$ we get

$$D_{*u}(Z^k Z^1 \Psi_{kl}(uZ)) = 0$$

This may be integrated with initial condition (4.13). We obtain

$$X^k X^1 \Psi_{kl}(X) = 0$$

We differentiate and get

$$2X^1 \Psi_{kl}(X) + X^m X^1 \nabla_{*k} \Psi_{ml}(X) = 0$$

Using (4.12) this becomes

$$X^1 \Psi_{kl}(X) + X^m X^1 \nabla_{*l} \Psi_{km}(X) = 0$$

Along straight lines (as above) this gives

$$uZ^1 \Psi_{kl}(uZ) + u^2 Z^m D_{*u} \Psi_{km}(uZ) = 0$$

$$u^2 D_{*u} \left(\frac{1}{u} X^m \Psi_{km}(X) \right) + X^m \Psi_{km}(X) = 0$$

$$u D_{*u} (X^m \Psi_{km}(X)) = 0$$

We integrate with initial condition (4.13) and find

$$X^m \Psi_{km}(X) = 0 \quad \square$$

Proposition 4.4: Let $\Phi \in (T_{[2]}^0, T)$. Then there exists a unique $\Lambda \in (T_1^0, T)$ such that

$$\Lambda(0) = 0 \quad (4.14)$$

$$d_* \Lambda(0) = 0 \quad (4.15)$$

and, for all $X \in M_Z$,

$$X \cdot (\Phi + \mathbb{I}_* \Lambda) = 0 \quad (4.16)$$

Proof: We show that condition (4.16) yields an ordinary differential equation for Λ along the rays emerging from $0 \in M_Z$, which may be integrated using initial values (4.14) and (4.15) in order to give a unique solution: With proposition 4.3 we bring (4.16) in the equivalent form (iii). This gives along straight lines $u \mapsto uX$

$$D_{*u}^2 \Lambda_m(uX) = X^k X^l \nabla_{\{\star k} \Phi_{ml\}}(uX) \quad (4.17)$$

(4.13) and (4.15) give the initial condition

$$\nabla_* \Lambda(0) = \Phi(0) \quad (4.18)$$

(4.17), (4.14) and (4.18) together determine Λ uniquely, and it is clear from the construction and from the equivalence (i) $\Leftrightarrow$ (iii) proven in proposition 4.3 that this Λ satisfies (4.14), (4.15) and (4.16). $\square$

def. 4.1: We define two maps from $(T_{[2]}^{\circ}, T)$ to $(T_{[2]}^{\circ}, T)$ by

$$\mathcal{P} : \Phi \mapsto \Phi + \$* \Lambda \quad (4.19)$$

$$\mathcal{U} : \Phi \mapsto - \$* \Lambda \quad (4.20)$$

where Λ is determined by proposition 4.4.

Lemma 4.5: Let $\Phi \in (T_{[2]}^{\circ}, T)$. Then

$$\begin{aligned} \mathcal{U}[\Phi]_{km} &= \Phi_{km}(0) + 2 \int_0^1 (1-u) D_{*u} \Phi_{km}(uX) du + \\ &+ \int_0^1 u(1-u) X^n X^l (\nabla_{*kn} \Phi_{ml} - \nabla_{*km} \Phi_{ln} + \nabla_{*ml} \Phi_{kn}) du \end{aligned} \quad (4.21)$$

and

$$\mathcal{P}[\Phi]_{kl} = 4 X^m X^n \int_0^1 u(1-u) \nabla_{*[k[l} \Phi_{m]n]}(uX) du \quad (4.22)$$

Proof: In virtue of (4.20)

$$\mathcal{U}[\Phi]_{km} = \nabla_{*(k} \Lambda_{m)}$$

where Λ is the solution of (4.17) with initial conditions (4.14) and (4.18). This initial value problem is solved explicitly by application of Taylor's formula to

$$f(u) := \Lambda(uX)$$

We find

$$f'(u) = \nabla_{*X} \Lambda(uX)$$

and in view of (4.14) and (4.18)

$$f(0) = 0$$

$$f'(0) = X \cdot \Phi$$

Using (4.1) we get

$$\begin{aligned} \Lambda_m(X) &= X^k \Phi_{km}(0) + \\ &+ \int_0^1 (1-u) X^k X^l \nabla_* \{ {}^k \Phi_{ml} \} (uX) du \end{aligned}$$

Application of ∇_* and symmetrization gives (4.21).

In order to show (4.22) we use proposition 4.2 together with (4.19) and (4.16):

$$X^k \mathcal{P}[\Phi]_{kl}(X) = 0$$

yields (4.22) since

$$\nabla_* [k[lm] \wedge n] = 0 \quad \square$$

Corollary 4.6: Let $S_z \subset M_z$ be closed and star-shaped with respect to $0 \in M_z$ for all $z \in M$. Let furthermore $\Phi \in (T_{[2]}^0, T)$ such that for all $z \in M$

$$\text{supp} (\Phi | M_z) \cap S_z = \emptyset$$

Then for all $z \in M$

$$\text{supp} (\mathcal{P}[\Phi] | M_z) \cap S_z = \emptyset$$

$$\text{supp} (\mathcal{U}[\Phi] | M_z) \cap S_z = \emptyset \quad \square$$

In the following proposition we give a summary of all properties of the decomposition operators $\mathcal{P}$ and $\mathcal{U}$:

Proposition 4.7: $\mathcal{P}$ and $\mathcal{U}$ are continuous linear maps from $(T_{[2]}^{\circ}, T)$ to $(T_{[2]}^{\circ}, T)$ with the following properties:

$$\mathcal{P} + \mathcal{U} = 1 \quad (4.23)$$

$$\mathcal{P} \circ \mathcal{P} = \mathcal{P} , \quad \mathcal{U} \circ \mathcal{U} = \mathcal{U} \quad (4.24)$$

$$\mathcal{P} \circ \mathcal{U} = \mathcal{U} \circ \mathcal{P} = 0 \quad (4.25)$$

$$\mathcal{P}[\Phi] = \Phi \iff x \cdot \Phi(x) = 0 \quad (4.26)$$

$$\mathcal{U}[\Phi] = \Phi \iff \Phi = \Phi_{\star} \quad (4.27)$$

Proof: Continuity and linearity follow directly from Lemma 4.5.

(4.23), (4.24), (4.25), (4.26) and (4.27) are immediate consequences of def. 4.1, proposition 4.3 and proposition 4.4. $\square$

2.5. Decomposition of $\mathcal{F}_{[2]}^{\circ}$ with Respect to a Timelike Curve Contained in W

After the decomposition of $(T_{[2]}^{\circ}, T)$ we attack the problem of the decomposition of $\mathcal{F}_{[2]}^{\circ}$. Since we are in curved space the generalization of proposition 4.7 is valid only in normal neighbourhoods. This creates no problems in spacelike directions in virtue of the properties of M and W stated in section 2.1, but W is infinite in timelike directions whereas the radius of injectivity of the exponential map is generically finite (for estimates of the distance inbetween conjugate points along timelike geodesics see

for example [18]). Furthermore we want a description of the body at any particular instant of time. One solves these problems in the following way: One moves the pole of the normal coordinate system, in which the decomposition is done, along the timelike curve $l = z(s)$ and restricts the decomposition to each $\Sigma(s)$. This yields the required result but complicates the decomposition considerably.

Proposition 5.1: Let $z \in M$, $\beta \in \mathcal{E}_{[2]}^0$. In a normal neighbourhood of z the following two are equivalent:

$$(i) \quad \sigma^a(z, x) \beta_{ab}(x) = 0 \quad (5.1)$$

$$(ii) \quad \sigma^b(z, x) \sigma^c(z, x) \nabla_b \beta_{ac}(x) = 0 \quad (5.2)$$

$$\beta(z) = 0 \quad (5.3)$$

(This is analogous to (i) $\Leftrightarrow$ (iii) in proposition 4.3)

Proof: (i) $\Rightarrow$ (ii): Suppose (5.1). Then

$$\nabla_c (\sigma^a \beta_{ab}) = 0 = \sigma^a_{;c} \beta_{ab} + \sigma^a \nabla_c \beta_{ab} \quad (5.4)$$

Transvection with σ^b and σ^c respectively gives

$$\begin{aligned} \sigma^a \sigma^b \nabla_c \beta_{ab} &= 0 \\ \sigma^a \sigma^c \nabla_c \beta_{ab} &= 0 \end{aligned} \quad (5.5)$$

(5.5) yields directly (5.2). Passing to the coincidence limit $x \rightarrow z$ in (5.4) we obtain (5.3).

(ii) $\Rightarrow$ (i) : Suppose (5.2) and (5.3). Let $x(u)$ be an affinely parametrised geodesic with $x(0) = z$. We multiply (5.2) with σ^a .

This gives

$$\sigma^a \sigma^b \sigma^c \nabla_a \beta_{bc} = 0$$

Using

$$\sigma^a = u \dot{x}^a \quad (5.6)$$

we get

$$D_u (\dot{x}^a \dot{x}^b \beta_{ab}) = 0$$

This may be integrated with initial condition (5.3):

$$\dot{x}^a \dot{x}^b \beta_{ab} = 0$$

This implies

$$2 \sigma^b \sigma^c \beta_{bc} + \sigma^b \sigma^c \nabla_a \beta_{bc} = 0 \quad (5.7)$$

We insert (5.7) in (5.2) and get

$$D_u \left(\frac{1}{u} \beta_{ac} \sigma^c \right) = - \frac{1}{u^2} \sigma^b \sigma^c \beta_{bc}$$

or

$$u D_u (\beta_{ab} \sigma^b) + (\sigma^b_a - \delta^b_a) \beta_{bc} \sigma^c = 0 \quad (5.8)$$

Using equation (5.A1) of appendix 2.5A, (5.8) is seen to be equivalent with

$$D_u (H_k^a \beta_{ac} \sigma^c) = 0$$

Using again the initial condition (5.3) this is integrated to give

$$H_k^a \beta_{ac} \sigma^c = 0$$

Since H_k^a is nonsingular this concludes the proof. $\square$

Proposition 5.2: Let $\lambda \in \mathcal{E}_1^0$, $\varphi \in \mathcal{E}_{[2]}^0$.

Put

$$\beta := \varphi + \mathcal{L} \lambda \quad (5.9)$$

Then (5.2) is equivalent with

$$D_u^2 \lambda_a + R_{a\dot{x}\dot{x}}{}^d \lambda_d = \dot{x}^b \dot{x}^c \nabla_{[b} \varphi_{a c]} \quad (5.10)$$

along all affinely parametrized geodesics $x(u)$ emanating from a point $z \in M$, where $\dot{x} = \frac{dx}{du}$.

Proof: It is an easy consequence of the definition and the symmetries of the Riemann tensor, that for any covector field λ the Jacobi equation

$$D_u^2 \lambda_a + R_{a\dot{x}\dot{x}}{}^d \lambda_d = 0$$

satisfied along all affinely parametrized geodesics through z is equivalent with

$$\sigma^b{}_{(z,x)} \sigma^c{}_{(z,x)} \nabla_{[b} \lambda_{c]}(x) = 0$$

Application of this result to (5.9) completes the proof. $\square$

Lemma 5.3: Suppose (5.9). Then

$$\lambda(z) = 0 \quad (5.11)$$

$$d\lambda(z) = 0$$

and (5.3) together are equivalent with

$$\lambda(z) = 0 \quad (5.12a)$$

$$\nabla\lambda(z) = \varphi(z) \quad (5.12b)$$

□

Proposition 5.4: Given $\varphi \in \mathcal{E}_{[2]}^0$ and $z \in M$ there is a unique choice of λ in (5.9) such that (5.1) and (5.11) hold (compare proposition 4.4.1).

Proof: In virtue of propositions 5.1 and 5.2 and lemma 5.3, equation (5.1) and (5.9) together are equivalent with (5.10) and (5.12) together, according to the following scheme:

$$\begin{array}{llll} \text{prop. 5.1} & \left\{ \begin{array}{l} (5.2) \\ (5.3) \end{array} \right\} & \begin{array}{l} \text{prop. 5.2} \\ \text{Lemma 5.3} \end{array} & \begin{array}{l} (5.10) \\ (5.12) \end{array} \\ (5.1) \quad \langle \Longleftrightarrow \rangle & & & \\ \text{trivial} & & & \\ (5.11) \quad \langle \Longleftrightarrow \rangle & (5.11) & & \end{array}$$

But (5.10) can be integrated along all geodesics through z using the initial conditions (5.12). The 1-form field obtained in such a way satisfies trivially (5.10) and (5.12) and therefore (5.1) and (5.11). The uniqueness of λ is a consequence of the classical theorem on uniqueness and existence of solutions of ordinary differential equations. □

Proposition 5.5: For any $\varphi \in \mathcal{E}_{[2]}^0$ the decomposition according to proposition 5.4 and equation (5.9) defines a field $\lambda \in T_{0,1}^{0,0}M$ and a field $\beta \in T_{0,[2]}^{0,0}M$ in a neighbourhood of the diagonal in $M \times M$.

Proof: According to proposition 5.4 one gets a decomposition of form (5.9) in the neighbourhood of any point $z \in M$, i.e.

$$\varphi(x) = \beta(z, x) - \oint_x \lambda(z, x) \quad (5.13)$$

The differentiability of β and λ with respect to z is a consequence of the dependence of solutions of ordinary differential equations on the data. $\square$

So far we have decomposed $\mathcal{E}_{[2]}^0$ in the neighbourhood of an arbitrary point z . This is somehow a generalization of section 2.4 and terminates the first part of our decomposition. Next we investigate the dependence of β and λ on s when we move z along the representative worldline $l = z(s)$.

Lemma 5.6: $\partial_s \lambda = \dot{z} \cdot \nabla_z \lambda$ obeys the equation

$$D_u^2 \partial_s \lambda_a + R_{a\dot{x}\dot{x}}^d \partial_s \lambda_d = 2u^{-2} \sigma^b \sigma^c \dot{z}^k \nabla_{[b} \beta_{ac]} \quad (5.14)$$

along any affinely parametrized geodesic $x(u)$ with $x(0) = z$.

Proof: (5.13) differentiated with respect to s yields

$$\partial_s \beta - \oint_x \partial_s \lambda = 0.$$

This is, for fixed z , just equation (5.9) if one substitutes

$$(\beta, \varphi, \lambda) \longmapsto (0, -\partial_s \beta, \partial_s \lambda)$$

Since (5.1) holds now trivially it is a consequence of properties (5.1) and (5.2) that

$$D_u^2 \partial_s \lambda_a + R_{a\dot{x}\dot{x}}^d \partial_s \lambda_d = -\dot{x}^b \dot{x}^c \nabla_{[b} \partial_s \beta_{ac]} \quad (5.15)$$

Differentiating (5.2) with respect to s we get

$$\sigma^b \sigma^c \nabla_{[b} \partial_s \beta_{ac]} + 2\sigma^b \sigma^c \dot{z}^k \nabla_{[b} \beta_{ac]} = 0 \quad (5.16)$$

Using (5.6) substitution of (5.16) into (5.15) gives (5.14). $\square$

Lemma 5.7: The initial conditions necessary to determine $\partial_s \lambda$ from (5.14) are

$$\partial_s \lambda_m = - \dot{z}^n \varphi_{nm} \quad (5.17a)$$

$$\nabla_m \partial_s \lambda_n = - 2 \dot{z}^r \nabla_{[m} \varphi_{n]r} \quad (5.17b)$$

where the limit $x \rightarrow z$ is taken after the differentiations have been performed, i.e. $\nabla_m \partial_s \lambda_n$ is the coincidence limit of $\nabla_a \partial_s \lambda_b$.

Proof: Differentiation of (5.12) gives (5.17) as follows:

(1) We differentiate (5.12a). This yields

$$\partial_s \lambda(z(s), z(s)) = \langle \partial_s \lambda \rangle + \langle \dot{z} \cdot \nabla_x \lambda \rangle = 0$$

along $z(s)$. Using (5.12b) we get

$$\partial_s \lambda_m + \dot{z}^n \varphi_{nm} = 0$$

which is (5.17a).

(2) Differentiation of (5.12b) gives

$$\langle \nabla_x \partial_s \lambda \rangle + \langle \dot{z} \cdot \nabla_x \nabla_x \lambda \rangle = \nabla_{\dot{z}} \varphi \quad (5.18)$$

It is a consequence of (5.12a) that

$$\nabla_{[kl]} \lambda_m = 0$$

or

$$\nabla_{kl} \lambda_m = \nabla_{(kl)} \lambda_m$$

In virtue of this property we get in the limit $u \rightarrow 0$ from (5.10)

$$\nabla_{kl} \lambda_m = \nabla_{\{k} \varphi_{m\}l} \quad (5.19)$$

We insert (5.19) into (5.18) and get

$$\nabla_k \partial_s \lambda_m + \dot{z}^n \nabla_n \Phi_{mk} = \dot{z}^n \nabla_n \Phi_{mk}$$

After evaluation of "{ }" this is seen to be (5.17b) □

Using the preceding results we are in the position to construct our decomposition operators. From now on - unless otherwise stated - all tensor fields are in the spaces $\mathcal{F}_s^r$.

def. 5.1: We define a map $\hat{\omega}: \mathcal{E}_{[2]}^0 \rightarrow \mathcal{E}_1^0$ by

$$\hat{\omega}[\varphi](x) := \lambda(z(\tau(x)), x)$$

where λ is the solution of (5.10) with initial values (5.12). $\hat{\omega}$ is well defined as a consequence of propositions 5.4 and 5.5 and the geometrical considerations of section 2.1.

Proposition 5.8: Let

$$y := \text{Exp}_z(-t \nabla_z \sigma(z, x))$$

As usually indices $a, b, \dots$ refer to x and $k, l, \dots$ to $z(\tau(x))$.

Moreover we denote indices at y by $p, q, \dots$. Then

$$\begin{aligned} \hat{\omega}_a[\varphi](x) &= -H_a^k \sigma^1 \varphi_{kl} + \\ &+ \int_0^1 dt \, t^{-2} (1-t) H_a^p \sigma^q(z, y) \sigma^r(z, y) \nabla_{\{q} \varphi_{pr\}}. \end{aligned} \quad (5.20)$$

Proof: Appendix 2.5B. □

Corollary 5.9: $\hat{\omega}$ is linear. □

$F \cap W$ is compact, this implies that $F \cap W \subset \bigcup_{s_1 \leq s \leq s_2} \Sigma(s) \cap W$ for some $s_1, s_2 \in \mathbb{R}$. Therefore

$$\hat{F} \cap W \subset \{y \mid y = \text{Exp}_z [-t \nabla_z \sigma(z, x)], x \in \bigcup_{s_1 \leq s \leq s_2} \Sigma(s) \cap W,$$

$$z = z(\tau(x)), t \geq 1\} \cap W = \bigcup_{s_1 \leq s \leq s_2} \Sigma(s) \cap W$$

$\hat{F} \cap W$ is closed, $\bigcup_{s_1 \leq s \leq s_2} \Sigma(s) \cap W$ is compact, thus $\hat{F} \cap W$ is compact, i.e. $\hat{F} \in \mathcal{F}$.

$$(2) \text{supp}(\phi) \subset F \Rightarrow \text{supp}(\hat{\mathcal{M}}[\phi]) \subset \hat{F}:$$

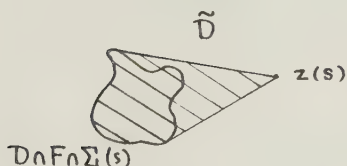
If $x \notin \hat{F}$ then $\gamma(z(\tau(x)), x) \cap F = \emptyset$. It follows, as in the proof of corollary 5.10, that $\hat{\mathcal{M}}[\phi](x) = 0$ for all $\phi \in \mathcal{F}_{[2]}^{\circ}(F)$. $\square$

Def. 5.2: We define $\mathcal{M}$ to be the restriction of $\hat{\mathcal{M}}$ to $\mathcal{F}_{[2]}^{\circ}$.

Proposition 5.12: $\mathcal{M}$ is a continuous linear map from $\mathcal{F}_{[2]}^{\circ}$ to $\mathcal{F}_1^{\circ}$.

Proof: In virtue of corollaries 5.9 and 5.11 $\mathcal{M}$ is a linear map from $\mathcal{F}_{[2]}^{\circ}$ to $\mathcal{F}_1^{\circ}$. As a further consequence of corollary 5.11 we may use proposition 1.3.11 and equation (5.20) in order to show continuity: Let $F \in \mathcal{F}$, $D \in \mathcal{F}^{\perp}$, $n \in \mathbb{N}$. We define

$$\tilde{D} := \bigcup_{x \in D \cap F} \gamma(z(\tau(x)), x)$$



(Fig. 5.2)

i.e. $\tilde{D}$ consists of all geodesic segments with one endpoint in $D \cap F$ and the other on $z(s)$ which are contained in some $\Sigma(s)$.
(Fig. 5.2).

Since $D \cap F$ is compact, $\tilde{D}$ is compact. Now it is an immediate consequence of equation (5.20) that for all $\varphi \in \mathcal{F}_2^0$

$$P_{D,n}(\mu[\varphi]) \leq C p_{D,n+1}(\varphi)$$

where C is a positive constant independent on φ . □

Def. 5.3: We define $\mu: \mathcal{F}_{[2]}^0 \rightarrow (\mathcal{E}_1^0)^{\mathbb{R}}$ by

$$\mu[\varphi](s, x) := \lambda(z(s), x)$$

where λ is the solution of (5.10) with initial values (5.12).

Remark: $(\mathcal{E}_1^0)^{\mathbb{R}}$ is an abuse of notation, since according to proposition 5.5 $x \mapsto \mu[\varphi](s, x)$ is well defined only in a normal neighbourhood of $z(s)$. As we shall need $x \mapsto \mu[\varphi](s, x)$ only in a neighbourhood of $\Sigma(s)$, which is by hypothesis (see section 2.1) contained in a normal neighbourhood of $z(s)$, the introduction of a new symbol for this space would increase complication without any real gain of precision.

Def. 5.4: Let λ be the same as in definition 5.3. We define the following maps:

$$\nu: \mathcal{F}_{[2]}^0 \rightarrow (\mathcal{E}_{[2]}^0)^{\mathbb{R}} \tag{5.22a}$$

$$\nu[\varphi](s; x) := - \$_x \lambda(z(s), x) \tag{5.22b}$$

$$\tau: \mathcal{F}_{[2]}^0 \rightarrow (\mathcal{E}_{[2]}^0)^{\mathbb{R}} \tag{5.23a}$$

$$\tau[\varphi](s; x) := \beta(z(s), x) := \phi(x) + \$_x \lambda(z(s), x) \tag{5.23b}$$

$$\vartheta: \mathcal{F}_{[2]}^0 \rightarrow (\mathcal{E}_1^0)^{\mathbb{R}} \tag{5.24a}$$

$$\partial[\varphi](s; x) = \partial_s \lambda(z(s), x) \quad (5.24b)$$

Let, furthermore, $\gamma : M \ni x \mapsto (\tau(x), x) \in \mathbb{R} \times M$.

Then we define $\underline{b}$, $\underline{\tau}$, $\underline{\partial}$ by

$$\underline{b}[\varphi] := b[\varphi] \circ \gamma \quad (5.22c)$$

$$\underline{\tau}[\varphi] := \tau[\varphi] \circ \gamma \quad (5.23c)$$

$$\underline{\partial}[\varphi] := \partial[\varphi] \circ \gamma \quad (5.24c)$$

Remark: Obviously

$$\underline{\lambda}[\varphi] = \lambda[\varphi] \circ \gamma$$

The above remark on notations like $(\mathcal{E}_1^0)^{\mathbb{R}}$ applies also to definition 5.4.

Proposition 5.13: Let $\psi(z, x)$ be the solution of (5.14) with zero initial data. Then

$$\psi_a(z, x) = 2 \int_0^1 t^{-2} (1-t) H_a^p \sigma^q(z, y) \sigma^r_k \dot{z}^k \cdot \nabla_{\{q\} \beta_{pr}} dt \quad (5.25)$$

where the conventions are the same as in proposition 5.8. The solution of (5.14) with initial conditions (5.17) is then

$$\begin{aligned} \partial_s \lambda_a(z(s), x) &= \psi_a(z(s), x) - \\ &- K_a^k(z(s), x) \dot{z}^1(s) \varphi_{k1}(z(s)) - \\ &- 2H_a^{[k}(z(s), x) \sigma^{1]}(z(s), x) \dot{z}^m(s) \nabla_k \varphi_{m1}(z(s)) \end{aligned}$$

Proof: Appendix 2.5B. $\square$

Remark: The singularity in the integral (5.25) is only apparent, since $\sigma^q \nabla_{[q} \varphi_{pr]}$ goes to zero as t^n , $n \geq 2$, in the limit $t \rightarrow 0$. This may be seen as follows: (5.1) implies the existence of a $h \in \mathcal{E}_{[2,2]}^0$ such that $\beta_{ab} = \sigma^d \sigma^e h_{adbe}$. This is a consequence of proposition 4.3: Let $B := \text{Exp}^* \beta$. Then (5.1) is equivalent to $X \cdot B = 0$. In virtue of proposition 4.3 this implies $B_{kl} = X^m X^n H_{kmnl}$. We apply Exp^{*-1} and get $\beta_{ab} = \sigma^d \sigma^e h_{adbe}$ with $h = \text{Exp}^{*-1} H$. Therefore all terms in $\nabla_x \sigma \cdot \nabla \beta$ contain at least two times $\nabla_x \sigma$ which is proportional to t . $\square$

Def. 5.5: We define $\mathcal{H}$ to be the subspace of $T_{0, [2]}^0(1 \times M)$ consisting of the fields satisfying

$$\sigma^b(z, x) \beta_{ab}(z, x) = 0$$

for all $z \in 1$ and all x in a normal neighbourhood of z . We endow $\mathcal{H}$ with the topology corresponding to the K -family of closed sets.

In order to simplify notation we write $\beta(s, x)$ for $\beta(z(s), x)$.

Lemma 5.14: $\mathcal{H}$ is a closed linear subspace of $\mathcal{E}_{[2]}^0(1 \times M)$

Proof: Let $\beta_n \xrightarrow{\mathcal{E}} \beta$, $(\beta_n)_{n \in \mathbb{N}} \subset \mathcal{H}$. This implies that $\nabla_x \sigma \cdot \beta_n \xrightarrow{\mathcal{E}} \nabla_x \sigma \cdot \beta$, but for all $n \in \mathbb{N}$, $\nabla_x \sigma \cdot \beta_n = 0$. Therefore $\nabla_x \sigma \cdot \beta = 0$. $\square$

Lemma 5.15: $\tau[\mathcal{F}_{[2]}^0] \subset \mathcal{H}$

Proof: Definition 5.4 and proposition 5.5. $\square$

Def. 5.6: Let ψ be as in proposition 5.13. We define a map

$\kappa: \mathcal{H} \rightarrow (\mathcal{E}_1^0)^{\mathbb{R}}$ (confer remark after def. 5.4) by

$$\kappa[\beta](s; x) := \psi(z(s), x) \quad (5.26a)$$

and

$$\underline{\kappa}[\beta] := \kappa[\beta] \circ \gamma \quad (5.26b)$$

Def. 5.7: $\mathcal{X} := \{\mu, \nu, \tau, \vartheta\}$

Lemma 5.16: Let $\varphi \in \mathcal{X}, \varphi \in \mathcal{F}_{[2]}^{\circ}, \beta \in \mathcal{H}$. Then $\varphi[\varphi]$ and $\mu[\beta]$ are smooth bitensor fields on $1 \times M$.

Proof: proposition 5.5. $\square$

Lemma 5.17: Let $F \in \mathcal{F}$. Then there exists a $\hat{F} \in \mathcal{F}$ such that for all $\varphi \in \mathcal{X}$ and all $\varphi \in \mathcal{F}_{[2]}^{\circ}, \varphi[\varphi] \in \mathcal{F}_s^{\circ}(\hat{F})$, where $s = 1$ or $s = [2]$. Furthermore for all $\beta \in \mathcal{H}$ such that $\beta \circ \gamma \in \mathcal{F}_{[2]}^{\circ}(F), \mu[\beta] \in \mathcal{F}_1^{\circ}(\hat{F})$.

Proof: Corollary 5.11 and analogous arguments using propositions 5.8 and 5.13. $\square$

Lemma 5.18: Let $S \in \mathcal{C}$ be such that $S \cap \Sigma(s)$ is star-shaped with respect to $z(s)$ for all $s \in \mathbb{R}$. Suppose $\varphi \in \mathcal{F}_{[2]}^{\circ}$ with $\text{supp}(\varphi) \cap S = \emptyset$. Then $\text{supp}(\varphi[\varphi]) \cap S = \emptyset$ for $\varphi \in \mathcal{X}$. Furthermore let $\beta \in \mathcal{H}$ with $\text{supp}(\beta \circ \gamma) \cap S = \emptyset$: This implies $\text{supp}(\mu[\beta]) \cap S = \emptyset$.

Proof: The same as the proof of corollary 5.10 using propositions 5.8 and 5.13. $\square$

Proposition 5.19: $\mu, \nu, \tau, \vartheta, \underline{\mu}, \underline{\nu}, \underline{\tau}, \underline{\vartheta}$ and $\underline{\mu}$ are continuous and linear.

Proof: As proposition 5.12. $\square$

Lemma 5.20: Let $\omega \in \mathcal{F}_1^{\circ}$ and

$$\begin{aligned} \mathcal{F}_a(s, x) &= K_a^k(z(s), x) \omega_k(z(s)) + \\ &+ H_a^k(z(s), x) \sigma^1(z(s), x) \nabla_{[k} \omega_{1]}(z(s)) \end{aligned} \quad (5.27)$$

Then

$$\mu[-\$ \omega] = \omega - \mathcal{F}$$

Proof: $\lambda[-\$ \omega](s, x)$ is the solution of the equation

$$D_u^2 \lambda_a + R_{a\dot{x}\dot{x}}^d \lambda_d = \dot{x}^b \dot{x}^c [\nabla_{(bc)} \omega_a + \nabla_{[ba]} \omega_c + \nabla_{[ca]} \omega_b] \quad (5.28)$$

along the geodesic $x(u)$ with $x(0) = z(s)$ and $x(1) = x$ for the initial values

$$\lambda(0) = 0 \quad (5.29a)$$

$$\nabla \lambda(0) = -\$ \omega(z(s)) \quad (5.29b)$$

Using the definition of the Riemann-tensor, (5.28) is transformed into

$$D_u^2 (\lambda_a - \omega_a) + R_{a\dot{x}\dot{x}}^d (\lambda_d - \omega_d) = 0$$

It is well-known (see for example [10]) that this has the following solution:

$$\lambda_a - \omega_a = K_a^k (\lambda_k - \omega_k) - H_a^k \sigma^1 (\nabla_1 \lambda_k - \nabla_1 \omega_k) \quad (5.30)$$

Substitution of (5.29) into (5.30) gives the final result. $\square$

Lemma 5.21: For $\varphi \in \mathcal{X}$ we define

$$\psi_{ab}(s, x) := (-u_k(s) \sigma^k(z(s), x)) \varphi_{ab}(s, x)$$

Then

$$\underline{1}[\psi] = 0 \quad (5.31)$$

Proof: We will show

$$\sigma^b \nabla_{[b} \psi_{ac]} = 0 ,$$

then (5.25) immediately gives (5.31): The Leibnitz rule yields

$$\sigma^b \nabla_{[b} \psi_{ac]} = \sigma^b \nabla_{[b} \varphi_{ac]} (-u_k \sigma^k) - \sigma^b (u_k \sigma^k)_{[b} \varphi_{ac]}$$

The first term of the right hand side vanishes on $\{z(s)\} \times \Sigma(s)$ since $u_k \sigma^k = 0$ for $x \in \Sigma(s)$. As to the second we observe that

$$\begin{aligned} \sigma^b (-u_k \sigma^k \{^b\} \varphi_{ac}) &= (-\sigma^k u_k) \varphi_{ac} + u_k \sigma^k {}_a \varphi_{cb} \sigma^b - \\ &\quad - u_k \sigma^k {}_c \sigma^b \varphi_{ba} \end{aligned} \quad (5.32)$$

The first term on the right hand side of (5.32) vanishes on $\{z(s)\} \times \Sigma(s)$ since $u_k \sigma^k = 0$, the second and the third by the definition of $\mathcal{H}$. $\square$

We summarize the properties of the maps $\varphi \in \mathcal{X}$, ψ , π and $\underline{\pi}$ in the following propositions:

Proposition 5.22: Let $\varphi \in \mathcal{X}$. Then φ is a continuous linear map

$$\varphi: \mathcal{F}_{[2]}^{\circ} \longrightarrow T_{0,s}^{\circ, \circ}(1 \times M) \quad (5.33a)$$

and ψ is a continuous linear map

$$\psi: \mathcal{F}_{[2]}^{\circ}(F) \rightarrow \mathcal{F}_s^{\circ}(\hat{F}) \quad (5.33b)$$

where $s = 1$ or $s = [2]$ and $\hat{F}$ is given by (5.21), i.e.

$$\hat{F} = \left\{ y \in M \mid y = \text{Exp}_z[-t \nabla_z \sigma(z, x)] , \ z = z(\tau(x)), \ t \geq 1, \ x \in F \right\}.$$

A fortiori ψ is a continuous linear map

$$\psi: \mathcal{F}_{[2]}^{\circ} \rightarrow \mathcal{F}_s^{\circ} \quad (5.33c)$$

and has the following support mapping property: Let $S \in \mathcal{C}$ such that $S \cap \Sigma(s)$ is star-shaped with respect to $z(s)$ for all $s \in \mathbb{R}$. Suppose $\varphi \in \mathcal{F}_{[2]}^{\circ}$ with $\text{supp}(\varphi) \cap S = \emptyset$. Then for all $\psi \in \mathcal{X}$, $\text{supp}(\psi[\varphi]) \cap S = \emptyset$. Furthermore π satisfies $\pi[\mathcal{F}_{[2]}^{\circ}] \subset \mathcal{H}$. The

following relations hold inbetween $\mathcal{M}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$:

$$\mathcal{B}[\varphi](s; x) = -\mathcal{S}_x \mathcal{M}[\varphi](s; x) \quad (5.33d)$$

$$\mathcal{B} + \mathcal{C} = 1_{\mathcal{T}_{12}^0} \quad (5.33e)$$

$$\mathcal{D}[\varphi](s; x) = \partial_s \mathcal{M}[\varphi](s; x) \quad (5.33f)$$

For $\omega \in \mathcal{T}_1^0$ and $\mathcal{F}$ defined by (5.27), i.e.

$$\mathcal{F}_a = K_a^k \omega_k + H_a^k \sigma^1 \nabla_{[k} \omega_{l]}$$

we find

$$\mathcal{M}[-\mathcal{S}\omega] = \omega - \mathcal{F} \quad (5.33g)$$

and

$$\mathcal{B}[\mathcal{S}\omega] = \mathcal{S}\omega + \frac{1}{2} \mathcal{L}_{\mathcal{F}} g \quad (5.33h)$$

$$\mathcal{C}[\mathcal{S}\omega] = -\frac{1}{2} \mathcal{L}_{\mathcal{F}} g \quad \square \quad (5.33i)$$

Proposition 5.23: $\mathcal{N}$ is a continuous linear map

$$\mathcal{N} : \mathcal{H} \longrightarrow T_{0,1}^{0,0} (1 \times M) \quad (5.34a)$$

and $\underline{\mathcal{N}}$ is a continuous linear map

$$\underline{\mathcal{N}} : \mathcal{H} \longrightarrow \mathcal{E}_1^0 \quad (5.34b)$$

such that for all $\beta \in \mathcal{H}$ satisfying $\beta \circ \nu \in \mathcal{T}_{12}^0(F)$, $\mathcal{N}[\beta] \in \mathcal{T}_1^0(\hat{F})$, where $\hat{F}$ is defined as above. If $S \in \mathcal{E}$ is such that $S \cap \Sigma(s)$ is star-shaped with respect to $z(s)$ for all $s \in \mathbb{R}$, then $S \cap \text{supp}(\beta \circ \nu) = \emptyset$ implies $\text{supp}(\mathcal{N}[\beta]) \cap S = \emptyset$. For $\varphi \in \mathcal{H}$ we have

$$\underline{\mathcal{N}}[(u \cdot \nabla_z \sigma) \varphi] = 0 \quad (5.34c)$$

and for $\mathfrak{f}^a(s, x) := K^a_k A^k(s) + H^a_k \sigma_l^{kl}(s)$ we find

$$\begin{aligned} n^a[\mathfrak{f}_x] &= \partial_s \mathfrak{f}^a - K^a_k D_s A^k - H^a_k \sigma_l^{kl} D_s B^{kl} \\ &= (D_s K^a_k) A^k + [D_s (H^a_k \sigma_l^{kl})] B^{kl} \end{aligned} \quad (5.34d)$$

n satisfies the following relation inbetween τ , ϑ and n

$$\vartheta_a[\varphi] = (n \circ \tau)_a[\varphi] - K_a^{kl} \varphi_{kl} - 2H_a^{kl} \sigma_l^{lm} \nabla_k \varphi_{lm}. \quad (5.34e)$$

Proof: The only thing which remains to prove is (5.34d): In view of (5.16) and (5.14), $n[\beta]$ is solution of

$$D_u^2 \psi_a + R_{a\dot{x}\dot{x}}^d \psi_d = -\dot{x}^b \dot{x}^c \nabla_{[b} \partial_s \beta_{ac]}$$

for zero initial data at $x = z(s)$. Substituting $\beta = \mathfrak{f}_x$ we find after manipulations similar to those in Lemma 5.20

$$D_u^2 (\psi_a - \partial_s \mathfrak{f}_a) + R_{a\dot{x}\dot{x}}^d (\psi_d - \partial_s \mathfrak{f}_d) = 0$$

wherefrom we deduce (5.34d). $\square$

Appendix A of Section 2.5: Derivation of a Bitensor Formula

Proposition 5.A1: Let $x(u)$ be a geodesic, $x(0) = z$, u an affine parameter. Then

$$\sigma^a_c(z, x) - \delta^a_c = -u \sigma^k_c D_u H^a_k. \quad (5.A1)$$

Proof: Let $x(u, v)$ be a 1-parameter family of geodesics, v being the family parameter and u an affine parameter along each geodesic. Furthermore let

$$x(0, v) = z \text{ for all } v \quad (5.A2a)$$

$$v^a := \frac{\partial}{\partial v} x^a \quad (5.A2b)$$

$$\dot{x}^a := \frac{\partial}{\partial u} x^a. \quad (5.A2c)$$

Then it is a simple consequence of the symmetry of the connection that

$$D_u v^a = D_v \dot{x}^a = \frac{1}{u} \sigma^a_b v^b. \quad (5.A3)$$

It follows from (5.A2a) that

$$v^k = 0.$$

Thus

$$v^a = u H^a_k \dot{v}^k \quad (5.A4)$$

and

$$D_u v^a = H^a_k \dot{v}^k + u D_u H^a_k \dot{v}^k. \quad (5.A5)$$

Inserting (5.A5) into (5.A3), using (5.A4) and the arbitrariness of $\dot{v}^k$, we obtain

$$\sigma^a_b H^b_k = H^a_k + u D_u H^a_k$$

Transvection with σ^k_c yields (5.A1). $\square$

Appendix B of Section 2.5: Solution of the Inhomogeneous Jacobi Equation

Proposition 5.B1:

The solution of the initial value problem

$$\dot{x}^\alpha(t) + A^\alpha_\beta(t) x^\beta(t) = 0 \quad (5B.1)$$

$$x^k(t_0) = x^k_0$$

is

$$x^\alpha(t) = R^\alpha_k(t_0, t) x^k_0$$

where R is solution of

$$\frac{d}{dt} R^\alpha_k(t_0, t) + A^\alpha_\beta(t) R^\beta_k(t_0, t) = 0$$

$$R^\alpha_k(t_0, t_0) = \delta^\alpha_k.$$

Proof: e.g. [1] .



Def. 5.B1: R is called the resolvent of (5.B1).

Proposition 5.B2:

$$R^\alpha_k(t_0, t) R^k_p(t_1, t_0) = R^\alpha_p(t_1, t)$$

$$R^\alpha_k(t_0, t) R^k_\beta(t, t_0) = \delta^\alpha_\beta$$

$$R^k_\alpha(t, t_0) R^\alpha_\lambda(t_0, t) = \delta^k_\lambda$$

Proof: e.g. [1] .



Proposition 5B.3: (Variation of Constant)

The solution of

$$\dot{x}^\alpha(t) + A^\alpha_\beta(t) x^\beta(t) = B^\alpha(t)$$

$$x^\alpha(t_0) = x^\alpha_0$$

is

$$X^{\alpha}(t) = R^{\alpha}_{\mu}(t_0, t) X^{\mu}_0 + \int_{t_0}^t R^{\alpha}_{\rho}(t', t) B^{\rho}(t') dt' . \quad (5.B2)$$

Proof: e.g. [1] .

□

Def. 5B.2: The inhomogeneous Jacobi equation is the following ordinary differential equation along the geodesics emerging from a point $z \in M$

$$D_t^2 \xi^a + R^a_{bcd} \dot{x}^b \dot{x}^c \xi^d = f^a \quad (5.B3)$$

where t is an affine parameter.

Proposition 5B.4: The solution of (5.B3) with initial conditions

$$\xi(z) = A$$

$$D_t \xi(z) = B$$

is

$$\begin{aligned} \xi^a(x(t)) &= K^a_k(z, x(t)) A^k + (t-t_0) H^a_k B^k + \\ &+ \int_{t_0}^t (t-t') H^a_q(x(t'), x(t)) f^q(x(t')) dt' . \end{aligned} \quad (5.B4)$$

Proof: The solution of the homogeneous equation is

$$\xi^a = K^a_k A^k + L^a_k B^k$$

where

$$L^a_k := (t-t_0) H^a_k .$$

In order to make the result of proposition 5B.3 applicable we transform our problem as follows:

$$x^{\alpha}(t) := \begin{pmatrix} \lambda^a \\ D_t \lambda^a \end{pmatrix} (t)$$

where $\alpha = 1 \dots 8$, $\dot{x}^a = \lambda^a$, $x^{a+4} = D_t \lambda^a$,

$$A_{\beta}^{\alpha}(t) := \begin{pmatrix} 0 & -\delta_d^a \\ R_{\dot{x}\dot{x}d}^a & 0 \end{pmatrix} (x(t))$$

and

$$R_{\chi}^{\alpha}(t_0, t) := \begin{pmatrix} K_k^a & L_k^a \\ D_t K_k^a & D_t L_k^a \end{pmatrix} (z, x(t))$$

Inserting this in (5.B2) yields

$$\begin{pmatrix} \mathfrak{f}^a \\ D_t \mathfrak{f}^a \end{pmatrix} = \begin{pmatrix} K_k^a & L_k^a \\ D_t K_k^a & D_t L_k^a \end{pmatrix} \begin{pmatrix} A^k \\ B^k \end{pmatrix} + \int_{t_0}^t \begin{pmatrix} K_q^a & L_q^a \\ D_t K_q^a & D_t L_q^a \end{pmatrix} \begin{pmatrix} 0 \\ f^q \end{pmatrix} dt' \quad (5.B5)$$

Taking the first component of (5.B5) gives (5.B4). $\square$

Corollary 5B.5: The solution of the inhomogeneous Jacobi equation with initial values $\mathfrak{f}_k$ and $\mathfrak{f}_{k;1}$ is, in a normal neighbourhood of z ,

$$\mathfrak{f}_a = K_a^k \mathfrak{f}_k - H_a^k \sigma^1 \mathfrak{f}_{k;1} + \int_0^1 (1-t) H_a^q f_q dt$$

where the tensor index q corresponds to the points

$$y(t) := \text{Exp}_z \left\{ -t \nabla_z \sigma(z, x) \right\} \quad \square$$

3. MOMENTUM, ANGULAR MOMENTUM AND SKELETON OF A DIVERGENCE-FREE DISTRIBUTION IN $(\mathcal{T}_{[2]}^0)'$.

3.1 Definitions

In this section we define momentum, angular momentum and skeleton of a distribution $T \in (\mathcal{T}_{[2]}^0)'$ satisfying $\nabla \cdot T = 0$. Furthermore we define a subspace of $(\mathcal{T}_{[2]}^0)'$ of those distributions which are admissible for the construction of a "reasonable" skeleton.

Def. 1.1: Let $T \in (\mathcal{T}_{[2]}^0)'$. Then we define

(1) the momentum of T with respect to $(1, u)$

$$P := \int_{\Sigma} T \cdot \partial \tau \, K \quad (1.1)$$

(2) the angular momentum of T with respect to $(1, u)$

$$S := \int_{\Sigma} T \cdot \partial \tau \, H \wedge \nabla_z \sigma \quad (1.2)$$

Remark: If T is a smooth tensor field, then P and S are smooth tensor fields along l given by

$$P^k(s) = \int_{\Sigma(s)} T^{ab} \partial_a \tau \, K_b^k \, w^c ds_c = \int_{\Sigma(s)} K_a^k T^{ab} \, ds_b$$

$$S^{kl}(s) = 2 \int_{\Sigma(s)} T^{ab} \partial_a \tau \, H_b^{[k} \sigma^{l]} w^c \, ds_c =$$

$$= 2 \int_{\Sigma(s)} H_a^{[k} \sigma^{l]} T^{ab} \, ds_b$$

as originally defined by Dixon [6].

Def. 1.2: $\mathcal{D}_s^r(1)$ denotes the $(\mathcal{D}_s^r)$ test fields over the family of compact sets on l . $\mathcal{D}_0^0(1) \cong \mathcal{D}(\mathbb{R})$.

Lemma 1.1: $P \in (\mathcal{D}_1^0(1))'$ and $S \in (\mathcal{D}_{[1, \eta]}^0(1))'$.

Proof: Proposition 2.3.6; the tensorial character of K and $H \wedge \nabla_z \sigma$ at z does not change the argument. $\square$

Def. 1.3: We define $Q : (\mathcal{F}_{[2]}^0)' \times \mathcal{H} \rightarrow (\mathcal{D}(\mathbb{R}))'$ by

$$Q[T, \beta] := \int_{\Sigma} T \beta \cdot \nu - \int_{\Sigma} T \cdot \partial \tau \cdot \underline{n}[\beta] \quad (1.3)$$

Lemma 1.2: Q is bilinear and partially continuous.

Proof: Propositions 2.3.7 and 2.5.23. $\square$

Def. 1.4: For fixed W we call T an admissible energy-momentum distribution with respect to $\Sigma = \{\Sigma(s) \mid s \in \mathbb{R}\}$ iff

$$(E \ 1) \quad T \in (\mathcal{F}_{[2]}^0)'$$

$$(E \ 2) \quad \text{supp}(T) \subset W$$

$$(E \ 3) \quad \nabla T = 0$$

$$(E \ 4) \quad T \text{ is restrictable to } \Sigma.$$

We denote the space of all admissible energy-momentum distributions for a fixed W by $\mathcal{M}_{W, \Sigma}$. In view of the remark after definition 2.1.1 we shall simply write $\mathcal{M}_{\Sigma}$.

Lemma 1.3: Let $T \in \mathcal{M}_{\Sigma}$. Then P and S are smooth vector (bivector) fields along Σ . $\square$

Lemma 1.4: Let $T \in \mathcal{M}_{\Sigma}$. Then for all $s \in \mathbb{R}$ exists a $t_s \in \mathcal{H}'$ such that for all $f \in \mathcal{D}(\mathbb{R})$

$$\langle Q[T, \beta] \mid f \rangle = \int ds \ t_s[\beta] \ f(s) \quad (1.4)$$

For all $\beta \in \mathcal{H}$ the map $s \mapsto t_s[\beta]$ is in $\mathcal{E}(\mathbb{R})$. Furthermore we find

$$\text{supp}(t_s) = \{z(s)\} \quad \times \quad (W \cap \Sigma(s))$$

Proof: Inserting (1.3) into (1.4) we find that

$$\langle Q[\tau, \beta] | f \rangle = \\ = \langle \int_{\Sigma} T \beta \cdot \gamma | f \rangle - \langle \int_{\Sigma} T \partial \tau \otimes \underline{u}[\beta] | f \rangle$$

Since T is restrictable, $\beta \cdot \gamma \in \mathcal{E}_{[2]}^0$ and $\partial \tau \otimes \underline{u}[\beta] \in \mathcal{E}_{[2]}^0$ imply that there exists a function in $\mathcal{E}(\mathbb{R})$ depending on β , which we shall denote by $s \mapsto t_s[\beta]$, namely

$$t_s[\beta] = \int_{\Sigma(s)} T \beta \cdot \gamma - \int_{\Sigma(s)} T \partial \tau \otimes \underline{u}[\beta] \quad (1.5)$$

in the notation of definition 2.3.3. It is trivial that $\beta \mapsto \beta \cdot \gamma$ and $\psi \mapsto \partial \tau \otimes \underline{u}[\psi]$ are continuous and linear. Therefore it follows from proposition 2.5.19 and the restrictability of T ((E 4) and (CF 2)) that $\beta \mapsto t_s[\beta]$ is continuous. Since $t_s[\beta] \in \mathbb{R}$, it is a distribution in $\mathcal{H}'$. Finally we choose a $\beta \in \mathcal{H}$ such that $\beta = 0$ in a neighbourhood of $\{z(s)\} \times (W \cap \Sigma(s))$. This implies that $\beta \cdot \gamma = 0$ in a neighbourhood S of $W \cap \Sigma(s)$. Furthermore W is star-shaped with respect to $z(s)$. Therefore we may suppose that S satisfies the conditions of Lemma 2.5.18. Hence $\text{supp}(\partial \tau \otimes \underline{u}[\beta]) \cap S = \emptyset$. Proposition 2.3.12 shows now that $t_s[\beta] = 0$. $\square$

Def. 1.5: Let $T \in \mathcal{M}_{\Sigma}$, $\Psi \in (T_{[2]}^0, T)$. We define the skeleton $\hat{T}$ of T as follows:

$$\hat{T} : M \times (T_{[2]}^0, T) \longrightarrow \mathbb{R} \quad (1.6a)$$

$$\langle \hat{T}_z, \Psi \rangle := 0, \quad z \notin I \quad (1.6b)$$

$$\langle \hat{T}_{z(s)}, \Psi \rangle := t_s \left[\text{Exp}_{z(s)}^{*-1} \mathcal{P}[\Psi] \right], \quad z(s) \in I \quad (1.6c)$$

3.2 Properties of the Skeleton

In this and the following section we show that momentum, angular momentum and skeleton of an admissible energy-momentum distribution satisfy the same conditions as derived for a smooth energy momentum tensor in [9] and [10].

Proposition 2.1: for all $z \in M$, $\hat{T}_z$ is a distribution of compact support. If $z \in I$ then

$$\text{supp}(\hat{T}_{z(s)}) \subset \text{Exp}_{z(s)}^{-1}(W \cap \Sigma(s))$$

Proof: Suppose $z \in I$ (for $z \notin I$ the assertion is trivial!):

Proposition 2.1 follows trivially from corollary 2.4.6, proposition 2.4.7 and Lemma 1.4. $\square$

Lemma 2.2: $\hat{T}_{z(s)}$ is completely determined by the restriction of T to $\Sigma(s)$, i.e. by the distribution $\varphi \mapsto \int_{\Sigma(s)} T \varphi$.

Proof: From definition 1.5 we know that $\hat{T}_{z(s)}$ is determined by t_s which, in turn, is given by (1.5) whence follows the assertion. $\square$

Proposition 2.3: $\nabla_* \cdot \hat{T}_z = 0$ for all $z \in M$.

Proof: Let $\Omega \in (T_1^0, T)$. Then

$$\langle \nabla_* \hat{T}_z, \Omega \rangle = \langle \hat{T}_z, \$_* \Omega \rangle$$

But $\mathcal{U}[\$_* \Omega] = \$_* \Omega$ and therefore $\mathcal{P}[\$_* \Omega] = 0$ (proposition 2.4.7), whence

$$\langle \nabla_* \hat{T}_z, \Omega \rangle = 0 \quad \square \quad (2.1)$$

Proposition 2.4: $(u_n X^n) X^{[i_1 \dots i_j] [k_1 \dots l]} = 0$

Proof: Let $H \in (T_{[2,2]}^0, T)$. Then

$$\langle (u_n X^n) X^{[i_1 \dots i_j] [k_1 \dots l]}_{z(s)}, H_{ijkl} \rangle = \langle \hat{T}_{z(s)}, \Psi \rangle$$

where

$$\Psi_{kl} = - (u_n x^n) x^i x^j H_{kilj}$$

Obviously $\Psi_{kl} x^l = 0$, whence

$$\mathcal{P}[\Psi] = \Psi$$

We define

$$\psi := \text{Exp}_{z(s)}^{*-1} \Psi$$

i.e.

$$\psi_{ab} = (u_n \sigma^n) \sigma^i \sigma^j \sigma_a^k \sigma_b^l H_{kilj} \cdot \text{Exp}^{-1}$$

Furthermore

$$\psi = (-u_n \sigma^n) \phi$$

with $\phi \in \mathcal{K}$ as a consequence of the symmetries of H . The factor $(-u_n \sigma^n)$ implies that $\psi \cdot \nu = 0$. From Lemma 2.5.21 we infer that $\mathcal{H}[\psi] = 0$. Therefore, since

$$\langle \hat{T}_{z(s)}, \Psi \rangle = \int_{\Sigma(s)} T \psi \cdot \nu - \int_{\Sigma(s)} T \partial \tau \otimes \mathcal{H}[\psi]$$

we find that

$$\langle \hat{T}_{z(s)}, \Psi \rangle = 0 \quad \square$$

Lemma 2.5: Let $\{\lambda_s\}_{s \in \mathbb{R}}$ be a smooth 1-parameter family of smooth 1-form fields. Furthermore let $\lambda \cdot \nu$ and $(\partial_s \lambda) \cdot \nu \in \mathcal{F}_1^0$ and $(\$_x \lambda) \cdot \nu \in \mathcal{F}_{[2]}^0$. Suppose that $L \in (\mathcal{F}_{[2]}^0)'$ and $\nabla \cdot L = 0$. Then

$$\langle L, (\$_x \lambda) \cdot \nu \rangle = \int \int_{\Sigma} L \cdot \partial \tau (\partial_s \lambda) \cdot \nu$$

Proof: In virtue of Lemma 2.3.11 we have

$$\int_{\Sigma} L \cdot \partial \tau \lambda \cdot \nu = \int_{\Sigma} L \cdot \partial \tau (\partial_s \lambda) \cdot \nu - \int_{\Sigma} L (\$_x \lambda) \cdot \nu$$

The third term, containing $\nabla \cdot L$, vanishes by hypothesis. Lemma 2.3.9 completes the proof. $\square$

Lemma 2.6: Suppose $T \in \mathcal{M}_\Sigma$ and $\Phi \in \mathcal{F}_{[2]}^0$. Then

$$\begin{aligned} \langle T, \Phi \rangle &= \int ds \int_{\Sigma(s)} T \underline{x}[\Phi] - \\ &\quad - \int ds \int_{\Sigma(s)} T \cdot \partial \tau \underline{\theta}[\Phi]. \end{aligned} \quad (2.2)$$

Proof: According to Lemma 2.3.8 we have

$$\langle T, \Phi \rangle = \int ds \int_{\Sigma(s)} T \Phi \quad (2.3)$$

From proposition 2.5.22, equation (2.5.33e), we know that

$$\Phi = \underline{\theta}[\Phi] + \underline{x}[\Phi] \quad (2.4)$$

We infer from Lemma 2.5.17 that

$$\lambda_s(x) = \lambda(s, x) := \lambda[\Phi](s; x)$$

satisfies the hypotheses of Lemma 2.5.

Using (2.5.33d) and (2.5.33f) we find from Lemma 2.5 that

$$\int_{\Sigma(s)} T \underline{x}[\Phi] = - \int_{\Sigma(s)} T \cdot \partial \tau \underline{\theta}[\Phi] \quad (2.5)$$

Inserting (2.4) in (2.3) and using (2.5) we get (2.2). $\square$

Lemma 2.7: Let $T \in \mathcal{M}_\Sigma$ and $\Phi \in \mathcal{F}_{[2]}^0$. Then

$$\begin{aligned} \langle T, \Phi \rangle &= \int ds \left\{ p^{k1} \Phi_{k1}(z(s)) + \right. \\ &\quad \left. + s^{k1} \nabla_k \Phi_{1m}(z(s)) + t_s \circ \tau[\Phi] \right\} \end{aligned} \quad (2.6)$$

Proof: We insert (2.5.34e) in (2.2) and find

$$\begin{aligned} \langle T, \Phi \rangle &= \int ds \left\{ \int_{\Sigma(s)} T \underline{x}[\Phi] - \right. \\ &\quad \left. - \int_{\Sigma(s)} \underline{\pi} \circ \tau[\Phi] + p^{k1} \Phi_{k1}(z(s)) + s^{k1} \nabla_k \Phi_{1m} \right\} \end{aligned}$$

Use of (1.5) yields (2.6). $\square$

Def. 2.1: For $\Lambda \in (T_1^0, T)$, $G \cdot \Lambda$ denotes the element of $(T_{[2]}^0, T)$ whose components are $\Lambda_m G_{kl}^m$.

Lemma 2.8: Let $\varphi \in \mathcal{F}_{[2]}^0$. Then

$$\text{Exp}^* \tau[\varphi] = \mathcal{P}[\text{Exp}^* \varphi + G \cdot \text{Exp}^* \mathfrak{u}[\varphi]] \quad (2.7)$$

Proof: Evaluating the identity

$$H_m^b H_k^a \nabla_a (H_l^c \sigma_c^m) = 0$$

we find

$$H_m^b G_{kl}^m = H_k^a \nabla_a H_l^b \quad (2.8)$$

$\Lambda = \text{Exp}^* \lambda$ takes in a local coordinate system the form

$$\Lambda_k = H_k^a \lambda_a$$

whence, using (2.8),

$$(G \cdot \text{Exp}^* \lambda)_{kl} = H_k^a (\nabla_a H_l^b) \lambda_b$$

Therefore

$$\$_*\Lambda = -G \cdot \Lambda + \text{Exp}^* \$_x \lambda \quad (2.9)$$

Now we put $\lambda := \mathfrak{u}[\varphi]$. Addsubtraction of $\text{Exp}^* \varphi$ on the right hand side of (2.9) yields, using (5.33d) and (5.33e),

$$\text{Exp}^* \tau[\varphi] = \text{Exp}^* \varphi + G \cdot \text{Exp}^* \mathfrak{u}[\varphi] + \$_* \text{Exp}^* \mathfrak{u}[\varphi] \quad (2.10)$$

In view of the equality

$$X = - \text{Exp}^* \nabla_z \sigma \quad (2.11)$$

Lemma 2.5.15 implies for all $X \in M_z$

$$X \cdot \text{Exp}^* \tau[\varphi](z, X) = 0 \quad (2.12)$$

Thus we deduce from proposition 2.4.7, equation (2.4.26)

$$\mathcal{P} [\text{Exp}^* \tau[\varphi]] = \text{Exp}^* \tau[\varphi] \quad (2.13)$$

From (2.4.25) and (2.4.27) we find that

$$\mathcal{P} [\$_* \text{Exp}^* \mathcal{M}[\varphi]] = 0 \quad (2.14)$$

Application of $\mathcal{P}$ to (2.10) and use of (2.13) and (2.14) yield (2.7) $\square$

Proposition 2.9: Let $T \in \mathcal{M}_\Sigma$, $\hat{T}$ the skeleton of T , $\varphi \in \mathcal{T}_{[2]}^\circ$. Then

$$\begin{aligned} \langle T, \varphi \rangle = & \int ds \left[p^k(s) \dot{z}^1(s) \varphi_{k1}(z(s)) + \right. \\ & + s^{k1}(s) \dot{z}^m(s) \nabla_k \varphi_{1m}(z(s)) + \\ & \left. + \langle \hat{T}_{z(s)}, \text{Exp}^* \varphi + G \cdot \text{Exp}^* \mathcal{M}[\varphi] \rangle \right] \end{aligned} \quad (2.15)$$

Proof: Lemma 2.8 and definition 1.5 yield

$$t_s \circ \tau[\varphi] = \langle \hat{T}_{z(s)}, \text{Exp}^* \varphi + G \cdot \text{Exp}^* \mathcal{M}[\varphi] \rangle \quad (2.16)$$

We insert (2.16) in (2.6) and get (2.15). $\square$

Proposition 2.10: Let $T \in \mathcal{M}_\Sigma$. Then $\{\hat{T}_z\}_{z \in 1}$, the skeleton of T , is smoothly composable with respect to $(1, u)$.

Proof: We have to verify (SC 1) and (SC 2) of definition 2.3.4.

(SC 1) has already been shown, see proposition 2.1. Now let

$\Phi \in (T_{[2]}^\circ, T)$. It follows from definition 1.5 and Lemma 1.4 that

$$s \mapsto \langle \hat{T}_{z(s)}, \Phi \rangle = t_s [\text{Exp}^{*-1}_{z(s)} \mathcal{P}[\Phi]]$$

is of class C^∞ , since (confer (2.11) and (2.12))

$$\text{Exp}^{*-1}_{z(s)} \mathcal{P}[\Phi] \in \mathcal{H} \quad \square$$

Appendix A of Section 3.2: Extended Skeleton, Moment Generating Function, Moments

Everything in this appendix refers to the tangent space over one arbitrary but fixed $z \in 1$. We therefore omit the argument z whenever there is no risk of confusion. As "basic quantity" we take the skeleton $\hat{T}$ of a $T \in \mathcal{M}_\Sigma$. Its basic properties are

$$\hat{T}^{[kl]} = 0 \quad (2.A1)$$

$$\nabla_* \hat{T} = 0 \quad (2.A2)$$

$\hat{T}$ has compact support contained in the hyperplane H orthogonal to u . (2.A3)

$$(u_i X^i)_X [\hat{T}^{kl}] [\mathcal{M}_X^n] = 0 \quad (2.A4)$$

Def. 2.A1: The extended skeleton of $T \in \mathcal{M}_\Sigma$ is defined by

$$\hat{T}_{\text{ex}}^{kl} = p^{(kl)} \delta - s^{m(kl)} \nabla_{*m} \delta + \hat{T}^{kl}, \quad (2.A5)$$

where δ denotes the distribution given by $\langle \delta, f \rangle = f(0)$ for all $f \in (T_0^0, T)$.

Lemma 2.A1: Let $T \in \mathcal{M}_\Sigma$, $\hat{T}_{\text{ex}}$ the extended skeleton of T , $\varphi \in \mathcal{F}_{12}^0$. Then

$$\langle T, \varphi \rangle = \int ds \langle \hat{T}_{\text{ex}}, \exp^* \varphi + G \cdot \text{Exp}^* u[\varphi] \rangle.$$

Proof: This follows from proposition 2.9 observing that

$$\text{Exp}_z^* \varphi(0) = \varphi(z),$$

$$- \langle \nabla_* \delta, \text{Exp}^* \varphi \rangle = \nabla_* \text{Exp}_z^* \varphi(0) = \nabla \varphi(0), \quad G(0) = 0,$$

$$\text{Exp}^* u[\varphi](0) = \mathcal{M}[\varphi](z, z) = 0. \quad \square$$

Def. 2.A2: The moment generating function $\hat{I}$ is defined to be the Fourier transform of $\hat{T}_{\text{ex}}$, i.e.

$$\hat{I}^{kl}(K) := \langle \hat{T}_{ex}^{kl}(X), e^{-iK \cdot X} \rangle \quad (2.A6)$$

Lemma 2.A2: $\hat{I}$ is an entire analytic function of exponential type which increases for real $K \rightarrow \infty$ not faster than a polynomial.

Proof: (2.A3) and the theorem of Paley-Wiener. $\square$

Def. 2.A3: The reduced moments of the first kind ("the I's") are defined by

$$I^{m_1 \dots m_\nu kl} := (i)^\nu \partial_{\star}^{m_1 \dots m_\nu} \hat{I}^{kl}(0) \quad (2.A7)$$

where $\partial_{\star}^m := \partial / \partial K_m$.

Lemma 2.A3: An alternative definition would be

$$I^{kl} = P^{(kl)} \quad (2.A8a)$$

$$I^{mkl} = S^{m(kl)} \quad (2.A8b)$$

$$I^{m_1 \dots m_\nu kl} = \langle \hat{T}^{kl}, X^{m_1} \dots X^{m_\nu} \rangle \quad (\nu \geq 2) \quad (2.A8c)$$

Proof: Equations (2.A5) and (2.A6). $\square$

Lemma 2.A4: The moments of the first kind have the following symmetry properties:

$$I^{m_1 \dots m_\nu kl} = I^{(m_1 \dots m_\nu)(kl)}, \quad (\nu \geq 2) \quad (2.A9)$$

$$I^{(m_1 \dots m_\nu k)l} = 0, \quad (\nu \geq 2) \quad (2.A10)$$

i.e. $I^{m_1 \dots m_\nu kl}$ has the irreducible symmetry $[\nu, 2]$.

Proof: (2.A9) can be seen directly from equation (2.A8c). (2.A2) implies

$$\nabla_{\star n}^{m_1 \dots m_\nu k n l} (X^{m_1} \dots X^{m_\nu} \hat{T}^{kl}) = (\nu+1) X^{m_1 \dots m_\nu k} \hat{T}^{kl}.$$

Integrated this gives (2.A10). This in turn trivially implies

$$I^{(m_1 \dots m_y k)1} - I^{(m_1 \dots m_y 1)k} = 0 \quad (y \geq 2) \quad (2.A11)$$

and

$$I^{(m_1 \dots m_y k1)} = 0 \quad (y \geq 2) \quad (2.A12)$$

According (2.A9) the I's have the reducible symmetries $[y] \otimes [2] = [y, 2] \oplus [y+1, 1] \oplus [y+2]$, but from (2.A11) and (2.A12) we see that the $[y+1, 1]$ and $[y+2]$ parts vanish. $\square$

Def. 2.A4: The reduced moments of the 2^{nd} kind ("the J's") are defined by

$$J^{m_1 \dots m_y i j k l} := I^{m_1 \dots m_y [i[kj]l]} \quad (y \geq 0) \quad (2.A13)$$

where antisymmetrization is taken separately over (i, j) and (k, l) .

Corollary 2.A5: The J's and I's are equivalent:

$$I^{m_1 \dots m_y k l} = 4 \frac{y-1}{y+1} J^{(m_1 \dots m_{y-1} | k | m_y) 1} \quad (y \geq 2). \quad \square \quad (2.A14)$$

Lemma 2.A6: The moments of the second kind have the following symmetry properties:

$$J^{m_1 \dots m_y i j k l} = J^{(m_1 \dots m_y) [ij][kl]} \quad (y \geq 0) \quad (2.A15)$$

$$J^{m_1 \dots m_y i [jkl]} = 0 \quad (y \geq 0) \quad (2.A16)$$

$$J^{m_1 \dots m_{y-1} [m_y i j] k l} = 0 \quad (y \geq 1) \quad (2.A17)$$

Proof: Direct application of Lemma 2.A4 to def. 2.A4. $\square$

Lemma 2.A7: The moments of the second kind obey the orthogonality relation

$$u_{m_1} J^{m_1 \dots m_y i j k l} = 0 \quad (y \geq 1) \quad (2.A18)$$

Proof:

$$u_{m_1} J^{m_1 \dots m_y, i j k l} = u_{m_1} \left\langle \hat{T}^{[j l], X^{m_1} \dots X^{m_y} i] X^{k]} \right\rangle$$

where antisymmetrization is taken over (j,i) and (l,k) separately.

Therefore

$$u_{m_1} J^{m_1 \dots m_y, i j k l} = \left\langle (u_{m_1} X^{m_1})_X \hat{T}^{[i j] X^{[l k]} X^{m_2} \dots X^{m_y}} \right\rangle \quad (2.A19)$$

for $y \geq 1$. But the r.h.s. of (2.A19) vanishes by (2.A4). $\square$

Lemma 2.A8: The moments of the second kind satisfy the following "derived" symmetry - orthogonality conditions:

$$J^{m_1 \dots m_y (i[jk] l)} = 0 \quad (2.A20)$$

$$J^{m_1 \dots m_{y-1} [m_y i] u k l} = 0 \quad (2.A21)$$

$$J^{u m_2 \dots m_{y-1} u m_y u} = 0 \quad \text{i.e. all contradictions} \quad (2.A22)$$

$$J^{m_2 \dots m_{y-1} u u m_y u} = 0 \quad \text{of } J \text{ with } u \otimes u \otimes u \quad (2.A23)$$

$$J^{m_2 \dots m_y u u u} = 0 \quad \text{vanish!} \quad (2.A24)$$

Proof: In virtue of (2.A15) equation (2.A16) may be written as

$$J^{m_1 \dots m_y, i j k l} - J^{m_1 \dots m_y, i k j l} + J^{m_1 \dots m_y, i l j k} = 0$$

Symmetrizing with respect to (i,l) makes the last term vanish as a consequence of (2.A15). This gives (2.A20). (2.A21) is obtained by a similar straightforward calculation from (2.A15), (2.A17) and (2.A18). (2.A22) is just a trivial consequence of (2.A18). (2.A23) and (2.A24) follow from (2.A15). $\square$

Lemma 2.A9: The moments of the first kind satisfy

$$u_{m_1} u_1 I^{m_1 \dots [m_y k] 1} = 0 \quad (y \geq 2) \quad (2.A25)$$

$$u_{m_1} u_k u_1 I^{m_1 \dots m_y k 1} = 0 \quad (y \geq 2) \quad (2.A26)$$

$$u_{m_1} u_{m_2} u_k I^{m_1 \dots m_y k 1} = 0 \quad (y \geq 2) \quad (2.A27)$$

$$u_{m_1} u_{m_2} u_{m_3} I^{m_1 \dots m_y k 1} = 0 \quad (y \geq 3) \quad (2.A28)$$

Proof: From (2.A14) and (2.A15) it follows that

$$\begin{aligned} I^{m_1 \dots m_{y+2} k 1} &= \frac{y+1}{(y+3)!} \sum_{\pi \in \mathcal{H}_{y+2}} J^{m_{\pi(1)} \dots m_{\pi(y+1)} k m_{\pi(y+2)} 1} = \\ &= \frac{1}{(y+2)(y+3)} \sum_{\substack{\kappa \neq l \\ \kappa, l}} J^{(m_1 \dots \hat{m}_\kappa \dots \hat{m}_l \dots m_{y+2}) m_\kappa k m_l 1} \end{aligned} \quad (2.A29)$$

where $\mathcal{H}_y$ denotes the symmetric group of $\{1, \dots, y\}$ and $\hat{m}_k$ indicates that m_k is to be omitted. (2.A15), (2.A18) and (2.A29) imply that

$$u_{m_1} I^{m_1 \dots m_{y+2} k 1} = - \frac{2}{(y+2)(y+3)} \sum_{2 \leq \kappa \leq y+2} J^{m_2 \dots \hat{m}_\kappa m_{y+2} k (u m_\kappa) 1}$$

Hence

$$\begin{aligned} u_{m_1} u_1 I^{m_1 \dots m_y k 1} &= \\ &= - [(y+2)(y+3)]^{-1} \sum_{2 \leq \kappa \leq y+2} \left[J^{m_2 \dots \hat{m}_\kappa m_{y+2} k u m_\kappa u} + \right. \\ &\quad \left. + J^{m_2 \dots \hat{m}_\kappa m_{y+2} k m_\kappa u u} \right] \end{aligned}$$

The second term on the r.h.s. vanishes in virtue of (2.A15).

Finally we get

$$\begin{aligned}
 & u_{m_1} u_l \cdot I^{m_1 \dots [m_{y+2} k] 1} = \\
 & = [(y+2)(y+3)]^{-1} \left[\sum_{2 \leq k \leq y+1} J^{m_2 \dots \hat{m}_k \dots [m_{y+2} k] u} u_{m_k} + \right. \\
 & \quad \left. + \sum_J^{m_2 \dots m_{y+1} u [km_{y+2}] u} \right]
 \end{aligned}$$

The first term vanishes as a consequence of (2.A21), the second in virtue of (2.A20). This proves (2.A25).

In consequence of (2.A14) any contraction of $u \otimes u \otimes u$ with I may be expressed as a sum of contractions of $u \otimes u \otimes u$ with J which all are zero by Lemma 2.A8, whence (2.A26), (2.A27) and (2.A28). $\square$

Lemma 2.A10: The above symmetry and orthogonality properties of the (reduced) moments correspond to the following structure of the moment generating function

$$\hat{I}^{kl}(K) = A^{kl}(\hat{K}) + K_u \partial_{\star}^{(k} B^{l)}(\hat{K}) + K_u^2 \partial_{\star}^{kl} C(\hat{K}) \quad (2.A30)$$

where

$$\gamma := g - u \otimes u,$$

$$\hat{K}_i := \gamma_i^1 K_1, \text{ i.e.}$$

$$K_i = \hat{K}_i + u_i K_u \quad (2.A31)$$

and

$$\partial_{\star}^{uAkl} = 0,$$

$$\partial_{\star}^{uB^1} = 0,$$

$$\partial_{\star}^{uC} = 0.$$

Proof: In virtue of Lemma 2.A2 and (2.A7) the power series expansion of $\hat{I}$ around 0 yields

$$\hat{I}^{kl}(K) = \sum_{\nu \geq 0} \frac{(-i)^\nu}{\nu!} K_{m_1} \dots K_{m_\nu} I^{m_1 \dots m_\nu, kl} \quad (2.A32)$$

Using (2.A32) we may transform the properties of the I 's in those of $\hat{I}$. It follows from (2.A10) that

$$K_k \hat{I}^{kl}(K) = K_k I^{kl} - i K_m K_k I^{mkl}$$

or

$$K_k \hat{I}^{kl}(K) = K_k P^{(k, l)} - \frac{i}{2} K_m K_k S^{ml} \hat{z}^k.$$

From the definition of $\hat{I}$, equation (2.A6), we deduce that

$$\partial_*^{imn} I^{kl}(K) = i \langle X^i X^m X^n \hat{T}_{ex}^{kl}, e^{-iKX} \rangle$$

Applying (2.A4) we obtain

$$\partial_*^u \partial_*^{[k[l] \hat{I}^m]n]} = 0 \quad (2.A33)$$

i.e.

$\partial_*^{[k[l] \hat{I}^m]n]}$ does not depend on K_u . Now let

$$\hat{\partial}_*^i := \gamma^i \partial_*^1.$$

Then

$$\partial_*^i = \hat{\partial}_*^i + u^i \partial_*^u.$$

If we insert (2.A31) in (2.A32) and use (2.A28) we find

$$\hat{I}^{kl}(K) = A_{(0)}^{kl}(\hat{K}) + K_u A_{(1)}^{kl}(\hat{K}) + K_u^2 A_{(2)}^{kl}(\hat{K}) \quad (2.A34)$$

where

$$A_{(0)}^{kl}(K) = \sum_{\nu \geq 0} \frac{(-i)^\nu}{\nu!} \hat{K}_{m_1} \dots \hat{K}_{m_\nu} I^{m_1 \dots m_\nu, kl} \quad (2.A35)$$

$$A_{(1)}^{kl}(K) = \sum_{\nu \geq 0} \frac{(-i)^{\nu+1}}{\nu!} \hat{K}_{m_1} \dots \hat{K}_{m_\nu} I^{m_1 \dots m_\nu, u kl} \quad (2.A36)$$

$$A_{(2)}^{kl}(K) = \sum_{\nu \geq 0} \frac{(-1)^{\nu+2}}{\nu!} \hat{K}_{m_1} \dots \hat{K}_{m_\nu} I^{m_1 \dots m_\nu} u^{kl} \quad (2.A37)$$

Differentiation of (2.A34) yields

$$\begin{aligned} \partial_{\star}^{kl} \hat{I}^{mn} &= \hat{\partial}_{\star}^{kl} A_{(0)}^{mn} + 2 u^{(k} \hat{\partial}_{\star}^{l)} A_{(1)}^{mn} + \\ &+ 2 u^k u^l A_{(2)}^{mn} + K_u \left(\hat{\partial}_{\star}^{kl} A_{(1)}^{mn} + 4 u^{(k} \hat{\partial}_{\star}^{l)} A_{(2)}^{mn} \right) + \\ &+ K_u^2 \hat{\partial}_{\star}^{kl} A_{(2)}^{mn} \end{aligned}$$

Applying (2.A33) we get

$$\hat{\partial}_{\star} [k [l_A^{mn}] n] = 0 \quad (2.A38)$$

$$\hat{\partial}_{\star} [k [l_A^{mn}] n] + 2 u^{[k} \hat{\partial}_{\star}^{l]} A_{(2)}^{mn}] + 2 u^{[k} \hat{\partial}_{\star}^{l]} [k_A^{n}] m] = 0 \quad (2.A39)$$

If $F = F \circ \gamma$ then $\partial_{\star} F = \hat{\partial}_{\star} F$.

Therefore $\hat{\partial}_{\star}$ in (2.A38) and (2.A39) may be replaced by $\partial_{\star}$. Now (2.A38) implies (Lemma 2.4.5)

$$\mathcal{P} [A_{(2)}] = 0.$$

Therefore (proposition 2.4.7)

$$A_{(2)}^{kl} = \partial_{\star}^{(k} c^{l)} \quad (2.A40)$$

where c is given by

$$\begin{aligned} c^m &= a^m + K_k b^{km} - K_k I^{uukm} + \\ &+ \int_0^1 (1-s) K_k K_l \partial_{\star}^{[k} A_{(2)}^{ml]}(sK) ds \end{aligned}$$

where a^m and $b^{km} = b^{[km]}$ are arbitrary constants appearing in the solution of equation (2.4.17) if one does not choose zero initial

values for c^m and $\partial_*^{[k} c^{m]}$. Using (2.A27), (2.A28) and (2.A37) we find that

$$u_m c^m = a^u + K_k b^{ku} \quad (2.A41)$$

$$\partial_*^u c^m = b^{um}$$

we choose

$$a^u = 0 \quad (2.A42)$$

$$b^{ku} = 0 \quad (2.A43)$$

such that

$$u_m c^m = 0 \quad (2.A44)$$

$$\partial_*^u c^m = 0 \quad (2.A45)$$

Let

$$D^{kl} := \partial_*^{[k} c^{l]} \quad (2.A46)$$

In virtue of (2.A44) and (2.A45) D satisfies

$$D^{uk} = 0 \quad (2.A47)$$

$$\partial_*^u D^{kl} = 0 \quad (2.A48)$$

Using (2.A40) and (2.A46) in (2.A39) we get

$$\partial_*^{[k} [l A_{(1)}^{m]n]} + u^{[k} \partial_*^{m]} D^{ln]} + u^{[l} \partial_*^{n]} D^{km]} = 0 \quad (2.A49)$$

Since $A_{(1)}$ and D do not depend upon K_u (see (2.A36) and (2.A48)) transvection of (2.A49) with u_m gives

$$\partial_*^k (\partial_*^{[l} A_{(1)}^{n]u} - D^{ln}) = 0 \quad (2.A50)$$

The first item in (2.A50) vanishes as a consequence of (2.A25) and (2.A36). Hence D is constant. We adjust the free part of b so that

$$D = 0$$

(2.A51)

which is possible because of (2.A43) and (2.A47). Hence, by (2.A46), there exists a scalar function C such that

$$c^k = \partial_*^k C$$

It follows from (2.A41), (2.A42) and (2.A43) that C does not depend on K_u . We plug (2.A51) in (2.A49). This yields

$$\partial_*^{[k[1} A_{(1)}^{m]n]} = 0$$

Therefore there exists a B such that

$$A_{(1)}^{mn} = \partial_*^{(m} B^{n)}$$

We solve this equation in the same way as above and find

$$B^m = e^m + K_k f^{km} - i K_k I^{ukm} + \int_0^1 (1-s) K_k K_l \partial_*^{[k} A_{(1)}^{m]l]}(sK) ds$$

where e^m and $f^{km} = f^{[km]}$ are constants. Using (2.A36) we get

$$\partial_*^{uB^m} = f^{um} - i I^{uum}.$$

We choose

$$f^{um} := i I^{uum}$$

such that

$$\partial_*^{uB^m} = 0,$$

which concludes the proof of (2.A30). $\square$

Lemma 2.A11: Equation (2.A30) implies that $\langle \hat{T}_{ex}, E \rangle$ depends on E only through the restrictions of E_{kl} , $\nabla_{*u}(X^1 E_{kl})$ and $\nabla_{*uu}(X^k X^1 E_{kl})$ to the hyperplane orthogonal to u .

Proof: Let $E \in (T_{[2]}^0, T)$, then the Parseval identity reads

$$(2\pi)^4 \langle \hat{T}_{ex}, E \rangle = \int DK \hat{T}^{kl}(K) \int DX E_{kl}(X) e^{iK \cdot X} \quad (2.A52)$$

We insert (2.A30) in (2.A52) and get

$$(2\pi)^4 \langle \hat{T}_{ex}, E \rangle = \int DK \left(A^{kl}(\hat{K}) + K_u \partial_{*}^{kl} B^1(\hat{K}) + K_u^2 \partial_{*}^{kl} C(\hat{K}) \right) \int DX E_{kl}(X) e^{iK \cdot X}$$

Using the standard formulae of Fourier transform (shuffling derivatives) and performing the integral in K_u and X^u one finds that the above assertion holds. $\square$

Lemma 2.A12: The skeleton obeys

$$(u_i X^i) X^{[j} \hat{T}^{k]1} u_l = 0 \quad (2.A53)$$

$$(u_i X^i) \hat{T}^{kl} u_k u_l = 0 \quad (2.A54)$$

Proof: (2.A25) is, in virtue of (2.A8c) equivalent with

$$\langle u_i X^i X^{[j} \hat{T}^{k]1} u_l, X^{m_1} \dots X^{m_v} \rangle = 0$$

for all $v \geq 0$, whence (2.A53). In the same way one proves (2.A54) from (2.A26). $\square$

Summary: The following properties are deduced using (2.A1), (2.A2), (2.A3) and (2.A4):

(1) The Skeleton:

$$(u_i X^i) X^{[j} \hat{T}^{k]1} u_l = 0$$

$$(u_i X^i) \hat{T}^{kl} u_k u_l = 0$$

$\langle \hat{T}, E \rangle$ depends only on the restriction of E_{kl} , $\nabla_{*u}(X^k E_{kl})$ and $\nabla_{*uu}(X^k X^l E_{kl})$ to the hyperplane $u_k X^k = 0$.

(2) The J 's:

$$J^{m_1 \dots m_y (i[jk]1)} = 0 \quad (y \geq 0)$$

$$u_j J^{m_1 \dots m_{y-1} [m_y i] j k 1} = 0 \quad (y \geq 1)$$

$$u_{m_1} u_1 J^{m_1 \dots m_{y-1} [m_y k] 1} = 0 \quad (y \geq 2)$$

(3) The I 's:

$$I^{(m_1 \dots m_y i j)} = 0 \quad (y \geq 1)$$

$$I^{m_1 (m_2 \dots m_y i j)} = 0 \quad (y \geq 2)$$

$$u_{m_1} u_1 I^{m_1 \dots m_{y-1} [m_y k] 1} = 0 \quad (y \geq 2)$$

$$u_{m_1} u_{m_2} u_{m_3} I^{m_1 \dots m_y k 1} = 0 \quad (y \geq 3)$$

$$u_{m_1} u_{m_2} u_1 I^{m_1 \dots m_y k 1} = 0 \quad (y \geq 2)$$

$$u_{m_1} u_k u_1 I^{m_1 \dots m_y k 1} = 0 \quad (y \geq 2)$$

(4) The Generating Function:

$$I^{k1} = A^{k1} + K_u \partial_{*}^{(k_B 1)} + K_u^2 \partial_{*}^{k1} C$$

with

$$\partial_{*}^u A^{k1} = 0$$

$$\partial_{*}^u B^k = 0$$

$$\partial_{*}^u C = 0$$

SKELETON	GENERATING FUNCTION	1st MOMENTS (I's)	2nd MOMENTS (J's)
$\hat{\Pi}[k, l] = 0$	$\hat{\Pi}[k, l] = 0$	$\sum_I m_1 \dots m_y k_1 \dots k_l \quad (m_1 \dots m_y) (k_1) \quad (l)$ $= 0 \quad (\nu \geq 0)$	$\sum_J m_1 \dots m_y j_1 \dots j_l \quad (m_1 \dots m_y) [j] [k, l]$ $= 0 \quad (\nu \geq 0)$
$\nabla_* \hat{\Pi} = 0$	$K_{k, l} \hat{\Pi}[k, l] =$ $= K_{k, l} I^{k, l} - l K_{m, l} I^{m, k, l}$	$\sum_I (m_1 \dots m_y k) l \quad = 0 \quad (\nu \geq 2)$ $I (m, k, l) = 0$	$\sum_J m_1 \dots m_y l [j, k, l] = 0 \quad (\nu \geq 0)$ $\sum_J m_1 \dots [m_y i, j] k_1 = 0 \quad (\nu \geq 1)$
$\hat{\Pi}$ has compact support	$\hat{\Pi}$ is an entire analytic function of exponential type increasing not faster than a polynomial for real $k \rightarrow \infty$.	The I's exist and are equivalent with $\hat{\Pi}$	The J's exist and are equivalent with $\hat{\Pi}$
$x^\nu X^{-j} \hat{\Pi}^{k, l} [l, X^m] = 0$	$\partial_*^u \hat{\Pi}^{k, l} [l, m, n] = 0$	$\sum_{m_1} m_1 \dots m_y l [j, k, l] = 0$ $(\nu \geq 1)$	$\sum_{m_1} m_1 \dots m_y i, j, k, l = 0$ $(\nu \geq 1)$

3.3 Dixon's Integral Laws of Motion

Now we have inquired about all properties of P , S and $\hat{T}$ apart from the consequence of the local law of motion, $\nabla \cdot T = 0$:

Momentum P and angular momentum S satisfy an evolution equation along l , the integral laws of motion. In the following we give an derivation of these evolution equations. The derivation is not changed by the distributional character of T , if $T \in \mathcal{M}_\Sigma$, but for reasons of completeness we repeat here a proof given in [10] .

The following definitions and conventions are chosen for the whole section: Let $\omega \in \mathcal{F}_1^0$. Furthermore let $A := \omega \circ z$, $B := -\frac{1}{2}(d\omega) \cdot z$ (i.e. $B_{kl} = \nabla_{[k} \omega_{l]}$). We define the two-point-tensor field $f \in T_{0,1}^{0,0}(l \times M)$ by

$$f_a(s, x) := K_a^k A_k + H_a^k \sigma^l B_{kl} \quad (3.1)$$

The Lie-derivative is extended to bi-tensor fields in the following way:

Let $X \in T_{0,1}^{0,0} M$ and $Y \in T_{s,q}^{r,p} M$. Let X_z be the vector field one obtains from the restriction of X to a fixed z . Then we denote by $\mathcal{L}_{X_z} Y$ the two-point-tensor field one obtains by application of $\mathcal{L}_{X_z}$ to Y simultaneously on each argument point, for example

$$\mathcal{L}_f \sigma^k(z(s), x) = \sigma^k_a f^a + \sigma^k_l f^l - f^k_{;l} \sigma^l \quad (3.2)$$

Inserting (3.1) into (3.2) we obtain

$$\mathcal{L}_f(\nabla_z \sigma) = 0 \quad (3.3)$$

Since Lie- and exterior derivative commute we find

$$\mathcal{L}_f(\nabla_z \nabla_x \sigma) = \mathcal{L}_f H = 0$$

Finally we get

$$\text{Exp}_Z^*(-\phi_x \mathfrak{f}) = \frac{1}{2} \mathcal{L}_{\mathfrak{f}} G \quad (3.4)$$

Lemma 3.1: Let $\{S_z\}_{z \in 1}$ be a smoothly composable family of distributions in $(\mathcal{E}_{[2]}^0(T_Z M))'$ with respect to $(1, u)$ and let $\Phi \in (T_{[2]}^0, T)$. Then there is a vector field $f[S, \Phi]$ and a bivector field $l[S, \Phi]$ along 1 such that

$$\langle S_z, \mathcal{L}_{\mathfrak{f}} \Phi \rangle = 2A_k f^k[S, \Phi] + B_{kl} l^{kl}[S, \Phi]$$

f and l are smooth, bilinear and uniquely determined by $S_z, \Phi, 1$ and u .

Proof: $\langle S_z, \mathcal{L}_{\mathfrak{f}} \Phi \rangle$ is linear in both arguments. Furthermore $\mathcal{L}_{\mathfrak{f}} \Phi$ is linear in $\mathfrak{f}$ and Φ and $\mathfrak{f}$ is the sum of terms linear in A and B . Since $A(z)$ and $B(z)$ are constants with respect to S_z , they may be taken out of the functional. So they define f and l uniquely and the bilinearity follows immediately. The smoothness of f and l follows now from the smooth composability of $\{S_z\}_{z \in 1}$. $\square$

Def. 3.1: We define the gravitational force F and torque L relative to $(1, u)$ acting on "the body" whose skeleton is $\hat{T}$ by

$$F := f[\hat{T}, G] \quad (3.5a)$$

$$L := l[\hat{T}, G] \quad (3.5b)$$

Remark: It can be shown that

$$F_k = \frac{1}{2} \langle \hat{T}^{1n}, \nabla_{k*} G_{1n} \rangle \quad (3.6a)$$

$$L_{kl} = 2 \langle \hat{T}^{mn}, G_{mn[k} x_{l]} \rangle \quad (3.6b)$$

(see, for example, [10])

In order to allow a physical interpretation we prove the following theorem, which gives a justification for the choice of the terms "force" and "torque" for F and L :

Theorem 3.2: The following two are equivalent:

- (i) Force and torque vanish for all admissible energy-momentum distributions (i.e. for all possible extended bodies contained in W)
- (ii) W is isometric to the closure of an open subset of a spacetime of constant curvature.

Proof: (i) $\Rightarrow$ (ii): Let $l \equiv z(s)$ be a curve, A a vector field and B a bivector field along l . We define

$$\mathcal{F}^a(z(s), x) := K^a_k A^k + H^a_k \sigma_l^{kl} B^{kl}$$

Then for an arbitrary $T \in \mathcal{M}_\Sigma$ whose skeleton is $\hat{T}$, the hypothesis implies

$$\langle \hat{T}, \mathcal{L}_{\mathcal{F}} G \rangle = 0 \quad (3.7)$$

By definition (see equation (3.3))

$$\mathcal{L}_{\mathcal{F}} X = 0$$

and therefore

$$X \cdot \mathcal{L}_{\mathcal{F}} G = \mathcal{L}_{\mathcal{F}} (X \cdot G) = \mathcal{L}_{\mathcal{F}} X = 0$$

Hence, in virtue of (2.4.23)

$$\mathcal{P}[\mathcal{L}_{\mathcal{F}} G] = \mathcal{L}_{\mathcal{F}} G$$

Furthermore, using (3.4), we find that (3.7) is equivalent to

$$t_s[\mathcal{L}_x \mathcal{F}] = 0$$

Inserting the definition of t_s (Lemma 1.4) we get for all s , A , B , l and T

$$\int_{\Sigma(s)} T_{\alpha\beta} \xi^\alpha - \int_{\Sigma(s)} T_{\alpha\beta} \partial_\alpha \tau \xi^\beta = 0$$

This holds a fortiori for point particles (see proposition 4.1.2 and theorem 4.2.6)¹⁾ whence for all s , A , B and all $x \in \Sigma(s) \cap W$ and all timelike future directed $n \in M_x$

$$n^a n^b \nabla_b \xi_a + n^a \partial_a \tau n^b \xi_b = 0$$

This implies that for all A , B and all $x \in W$

$$\nabla_{(a} \xi_{b)} + \partial_{(a} \tau \xi_{b)} = 0$$

Using (2.5.34d) this is easily seen to be equivalent to

$$A_k(s) \nabla_{(a} K^k_{b)} (z(\tau(x)), x) + \\ + B_{kl}(s) \nabla_{(a} (H^k_b) \sigma^l) (z(\tau(x)), x) = 0$$

for all A , B and x . This shows that there are 10 linear independent Killing vector fields on W , which is (ii).

(ii) $\Rightarrow$ (i) : See, for example, [10]. $\square$

Theorem 3.3: Let $T \in \mathcal{M}_\Sigma$. $\hat{T}$ denotes the skeleton of T , P the momentum, S the angular momentum of T with respect to $(1, u)$. Then as a consequence of the local law of motion, $\nabla \cdot T = 0$, P and S have to satisfy Dixon's integral laws of motion

$$F^k = D_S P^k + \frac{1}{2} S^{lm} \dot{z}^n R_{lmn}{}^k \quad (3.8a)$$

$$L^{kl} = D_S S^{kl} - 2 P^{[k} \dot{z}^{l]} \quad (3.8b)$$

¹⁾ There is no risk of a logical circle, since the present result is not used later on.

Proof: $\nabla \cdot T = 0$ implies

$$\langle T, \xi \omega \rangle = 0$$

Furthermore we have by Lemma 2.5.20

$$\eta[-\xi \omega] = \omega - \xi$$

and therefore in virtue of (2.5.33d) and (2.5.33e)

$$\tau[\xi \omega] = \xi$$

With Lemma 2.8 and equation (3.4) we find

$$\frac{1}{2} \xi_f^G = \mathcal{P} [\text{Exp}^*(-\xi \omega) + G \cdot \text{Exp}^* \eta[-\xi \omega]] \quad (3.10)$$

Observing that $\hat{T} \cdot \mathcal{P} = \hat{T}$ (proposition 2.3) we insert (3.9)

and (3.10) into (2.15) (proposition 2.9). This yields

$$0 = \int ds \left[P^{k\dot{z}1} \nabla_{(k} \omega_{1)} + S^{k\dot{z}1m} \nabla_{k(1} \omega_{m)} + \frac{1}{2} \langle \hat{T}_{z(s)}, \xi_f^G \rangle \right] \quad (3.11)$$

for all $\omega \in \mathcal{F}_1^0$.

Next we evaluate all items in (3.11) in terms of A and B. From the trivial identity

$$\nabla_k \omega_1 = \nabla_{(k} \omega_{1)} + \nabla_{[k} \omega_{1]}$$

it follows that

$$-\xi \omega(z(s)) = \nabla \omega(z(s)) + B(s) \quad (3.12)$$

whence

$$P^{k\dot{z}1} \nabla_{(k} \omega_{1)} = P^{k\dot{z}1} \nabla_{(k} A_{1)} + P^{[k\dot{z}1]} B_{k1} \quad (3.13)$$

Since $dd\omega = 0$ we have, taking $B_{mk} = \nabla_{[m} \omega_{k]}$ not only on 1 but on M,

$$\nabla_k B_{1m} + \nabla_1 B_{mk} + \nabla_m B_{k1} = 0$$

or

$$\nabla_{[k} B_{l]m} = -\frac{1}{2} \nabla_m B_{kl} \quad (3.14)$$

(3.12) and (3.14) yield

$$S^{kl} \dot{z}^m \nabla_{k(l} \omega_{m)} = S^{kl} \dot{z}^n \nabla_{kl} \omega_n + \frac{1}{2} S^{kl} D_s B_{kl} \quad (3.15)$$

Since S is skew, we get

$$S^{kl} \dot{z}^m \nabla_{k(l} \omega_{m)} = \frac{1}{2} S^{kl} D_s B_{kl} - \frac{1}{2} S^{kl} \dot{z}^m R_{klm}^{n} A_n \quad (3.16)$$

We insert (3.6), (3.13) and (3.16) in (3.11) and get after partial integration of the terms containing $D_s A$ and $D_s B$,

$$\begin{aligned} \int ds \left[A_k (-D_s P^k - \frac{1}{2} S^{lm} \dot{z}^n R_{lmn}^{k} + F^k) + \right. \\ \left. + B_{kl} (-\frac{1}{2} D_s S^{kl} + P^{[k} \dot{z}^{l]} + \frac{1}{2} L^{kl}) \right] = 0 \end{aligned} \quad (3.17)$$

Since A and B are arbitrary, (3.17) implies the vanishing of both brackets. $\square$

3.4 Reconstruction of a Distribution from a Skeleton

So far we have resolved the first of the two problems described in the introduction: We have seen that a momentum P , angular momentum S and skeleton $\hat{T}$ may be defined for every $T \in \mathcal{M}_\Sigma$, that $\hat{T}$ is a distribution and that P , S and $\hat{T}$ have for $T \in \mathcal{M}_\Sigma$ the same properties as for smooth energy momentum tensors. The question we are going to answer now is: Under what conditions can we construct a distribution $T \in \mathcal{M}_\Sigma$ if we are given P , S , $\hat{T}$? Dixon [10] has shown the following: If a smooth symmetric tensor field T satisfies (2.15) for some choice of P , S and $\hat{T}$ and if $\hat{T}$ satisfies (2.A2), then T necessarily satisfies the local law of motion. The structure of our reconstruction

theorem is different: We show that for any choice of P , S and $\hat{T}$ satisfying the integral laws of motion and the skeleton properties (2A.1) ... (2A.4), equation (2.15) defines an element of $\mathcal{M}_\Sigma$. In order to prove this theorem we need the following proposition:

Proposition 4.1: Let $\{\hat{T}_z\}_{z \in M}$ be a family of distributions satisfying the skeleton properties (2A.1) ... (2A.4). Furthermore let $\{\hat{T}_z\}$ be smoothly composable with respect to $(1/u)$. Let $H_z := \text{Exp}_z^{-1}(\Sigma(z))$. For a distribution Q on H_z we denote its extension to M_z by $\bar{Q}$, i.e. for all $\Phi \in \mathcal{E}(M_z)$

$$\langle \bar{Q}, \Phi \rangle = \langle Q, \Phi \circ \gamma \rangle$$

Then $\{\hat{T}_z\}$ has the following structure:

$$\hat{T}_z = \bar{Q}_z^{(0)} + \nabla_{*u} \bar{Q}_z^{(1)} + \nabla_{*u}^2 \bar{Q}_z^{(2)} \quad (4.1)$$

where $Q_z^{(q)}$, $q = 0, 1, 2$ are distributions on H_z with $\text{supp}(Q_z^{(q)}) \subset \text{supp}(\hat{T}_z)$. The decomposition is unique, the maps $\hat{T}_z \mapsto Q_z^{(q)}$ are continuous. The families of distributions $\{\bar{Q}_z^{(q)}\}$ are smoothly composable with respect to $(1, u)$.

Proof: $\text{supp}(\hat{T}_z) \subset H_z$ implies according to a standard theorem in the theory of distributions (see, for example [23], chapitre III, théorème XXXVI) that a unique locally finite decomposition with the above properties - excluded the smooth composability - exists, i.

$$\hat{T}_z = \sum_{q=0}^N \nabla_{*u}^q \bar{Q}_z^{(q)}$$

on any open subset $U \subset M_z$ with compact closure, N depending on U and z . Lemma 2.A11 implies that $N = 2$ for all U and z . It remains to show the smooth composability of $\{\bar{Q}_z^{(q)}\}$, $q = 0, 1, 2$. (SC 1) is trivial since $\text{supp}(\bar{Q}_z^{(q)}) \subset \text{supp}(\hat{T}_z)$. In order to show (SC 2) we observe that

$$\nabla_{*u}^q (u_k x^k)^p = \begin{cases} \frac{p!}{(p-q)!} (u_k x^k)^{p-q} & \text{for } q \leq p \\ 0 & \text{for } q > p \end{cases} \quad (4.2)$$

Let $\Psi \in (T_{[2]}^0, T)$. From (4.1) and (4.2) it follows that

$$\langle \hat{T}_{z(s)}, \frac{1}{2} (u_k x^k)^2 \Psi \rangle = \langle \bar{Q}_{z(s)}^{(2)}, \Psi \rangle$$

But $\frac{1}{2} (u_k x^k)^2 \Psi \in (T_{[2]}^0, T)$ and $\hat{T}$ is smoothly composable. Therefore $s \mapsto \langle \bar{Q}_{z(s)}^{(2)}, \Psi \rangle$ is smooth, i.e. $\{\bar{Q}_z^{(2)}\}$ is smoothly composable.

Analogously one finds

$$\langle \hat{T}_{z(s)}, (u_k x^k) \Psi \rangle = - \langle \bar{Q}_{z(s)}^{(1)}, \Psi \rangle + 2 \langle \bar{Q}_{z(s)}^{(2)}, \nabla_{*u} \Psi \rangle$$

Since $(u_k x^k) \Psi$ and $\nabla_{*u} \Psi \in (T_{[2]}^0, T)$ it follows, as above, that $s \mapsto \langle \bar{Q}_{z(s)}^{(1)}, \Psi \rangle$ is smooth, i.e. $\{\bar{Q}_z^{(1)}\}$ is smoothly composable. Finally

$$\langle \bar{Q}_{z(s)}^{(0)}, \Psi \rangle = \langle \hat{T}_{z(s)}, \Psi \rangle - \langle \bar{Q}_{z(s)}^{(1)}, \nabla_{*u} \Psi \rangle + \langle \bar{Q}_{z(s)}^{(2)}, \nabla_{*u}^2 \Psi \rangle$$

which implies that $\{\bar{Q}_z^{(0)}\}$ is smoothly composable. $\square$

Theorem 4.2: Let $l \cong z(s)$ be a smooth timelike curve, u a smooth timelike future directed unit vector field along l , P a smooth vector field along l , S a smooth bivector field along l and $\{\hat{T}_z \in (T_{[2]}^0, T)_z', z \in M\}$ a smoothly composable family with respect to (l, u) having the skeleton properties (2A1) ... (2A4). Furthermore let $\hat{T}_z = 0$ for $z \notin l$ and $\text{Exp}_{z(s)}(\text{supp}(\hat{T}_{z(s)})) \subset \Sigma(s) \cap W$. If P , S and $\hat{T}$ satisfy the integral laws of motion (3.8), then equation (2.15) defines an admissible energy-momentum distribution $T \in \mathcal{M}_\Sigma$.

Proof: (E 1): We have to show that $T \in (\mathcal{F}_{[2]}^0)'$. Let $\phi \in \mathcal{F}_{[2]}^0$. This implies that $\phi \circ z$ and $(\nabla \phi) \circ z$ have compact support. Therefore

$$\phi \mapsto \int_1 P^{kz1} \phi_{kl} \, ds$$

and

$$\varphi \mapsto \int_1 s^{klz^m} \nabla_k \varphi_{lm} ds$$

are well defined linear continuous functionals on $\mathcal{T}_{[2]}^0$. Next we observe that in view of Lemma 2.8 equation (2.15) may be written in the equivalent form

$$\langle T, \varphi \rangle = \int ds (p^{kz^l} \varphi_{kl} + s^{klz^m} \nabla_k \varphi_{lm} + \langle \hat{T}_{z(s)}, \text{Exp}_{z(s)}^* \tau[\varphi] \rangle) \quad (4.3)$$

$\text{Exp}_{z(s)}$ being a diffeomorphism on the hyperplane orthogonal to u^1 , $\hat{T}_{z(s)} \circ \text{Exp}_{z(s)}^*$ is a distribution with compact support contained in $\Sigma(s) \cap W$ by hypothesis. From the explicit expression of $\tau[\varphi]$ in terms of φ using (2.5.23b) and (2.5.20) one deduces that $\tau[\varphi](s; x) = 0$ whenever $\varphi = 0$ in a neighbourhood of $\gamma(z(s), x)$ ²⁾. Since $W \cap \Sigma(s)$ is star-shaped with respect to $z(s)$ we deduce that

$$\langle \hat{T}_{z(s)}, \text{Exp}_{z(s)}^* \tau[\varphi] \rangle = 0 \quad (4.4)$$

whenever $\text{supp}(\varphi) \cap W \cap \Sigma(s) = \emptyset$. Therefore the map $\check{\varphi}: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\check{\varphi}(s) := \langle \hat{T}_{z(s)}, \text{Exp}_{z(s)}^* \tau[\varphi] \rangle$$

has compact support. $\check{\varphi}$ is smooth in virtue of the smooth composability of $\hat{T}$ and the smoothness of $\text{Exp}_{z(s)} \tau[\varphi]$ (proposition 2.5.22). These properties and the linearity of $\varphi \mapsto \check{\varphi}(s)$ for all $s \in \mathbb{R}$ imply that

$$\varphi \mapsto \int ds \check{\varphi}(s)$$

is a well defined linear functional on $\mathcal{T}_{[2]}^0$. Now let $\varphi_n \xrightarrow{\mathcal{T}} 0$. In virtue of proposition 2.5.22 this implies $\tau[\varphi_n] \xrightarrow{\mathcal{E}} 0$ for fixed

1) see the remark following Lemma 2.1.1

2) confer proof of corollary 2.5.10

but arbitrary $s \in \mathbb{R}$. Since $\hat{T}_{Z(s)} \circ \text{Exp}_{Z(s)}^*$ is a distribution with compact support we deduce that $\check{\phi}_n \rightarrow 0$ for all $s \in \mathbb{R}$. We deduce from (4.4) that for all $F \in \mathcal{F}$ there exists a compact $I \subset \mathbb{R}$ such that $\text{supp}(\phi) \subset F$ implies $\text{supp}(\check{\phi}) \subset I$. $\phi_n \rightarrow 0$ implies that there is a $F \in \mathcal{F}$ such that for all $n \in \mathbb{N}$, $\text{supp}(\phi_n) \subset F$, i.e. that there exists a compact $I \subset \mathbb{R}$ such that for all $n \in \mathbb{N}$, $\text{supp}(\phi_n) \subset I$. This together with $\check{\phi}_n(s) \xrightarrow{\mathbb{R}} 0$ for all $s \in \mathbb{R}$ implies

$$\int ds \langle \hat{T}_{Z(s)}, \text{Exp}_{Z(s)}^* \epsilon[\phi] \rangle = \int_I \check{\phi}_n(s) ds \rightarrow 0$$

Proposition 1.4.1 completes the argument.

(E 2), $\text{supp}(T) \subset W$: Suppose $\phi \in \mathcal{F}_{[2]}^0(F)$, $F \cap W = \emptyset$. Then $1 \cap F = \emptyset$, i.e. the first two terms in (4.3) vanish. The third term is zero in view of (4.4), i.e.

$$\langle T, \phi \rangle = 0$$

for all $\phi \in \mathcal{F}_{[2]}^0$ such that $\text{supp}(\phi) \cap W = \emptyset$, i.e. $\text{supp}(T) \subset W$.

(E 3), $\nabla \cdot T = 0$: The proof consists of reading proposition 3.2 the other way round: Let ω and $\mathcal{F}$ be defined as in section 3.3.

Then

$$\text{Exp}^* \epsilon[\phi] = \frac{1}{2} \mathcal{L}_{\mathcal{F}}^G$$

This implies

$$\begin{aligned} \langle T, \mathcal{L}\omega \rangle &= - \int ds (P^{k1} \nabla_{(k} \omega_{1)} + \\ &+ S^{k1} \nabla_{k(1} \omega_{m)} + \frac{1}{2} \langle \hat{T}_{Z(s)}, \mathcal{L}_{\mathcal{F}}^G \rangle) \end{aligned}$$

Inserting the laws of motion (3.8) and using equations (3.13) and (3.15) we get for all $\omega \in \mathcal{T}_1^0$

$$\langle \nabla \cdot T, \omega \rangle = \langle T, \mathcal{L}\omega \rangle = 0$$

(E 4), T is restrictable: Let $\varphi \in \mathcal{E}_{[2]}^0$, $f \in \mathcal{D}(\mathbb{R})$. Then

$$\begin{aligned} \langle \int_T \varphi \mid f \rangle &= \int ds \left\{ p^{k1} \varphi_{k1}(z(s)) f(s) + \right. \\ &+ s^{k1} \nabla_k (\varphi_{1m} f \circ \tau)(z(s)) + \langle \hat{T}_{z(s)}, \text{Exp}_{z(s)}^* (\varphi \perp f) + \\ &\left. + G \cdot \text{Exp}_{z(s)}^* \mathcal{M}[\varphi \perp f] \right\} \end{aligned} \quad (4.5)$$

We observe

$$(\nabla_k (\varphi_{1m} f \circ \tau)) \circ z = (\nabla_k \varphi_{1m}) \circ z f + \varphi_{1m} \circ z u_k Df \quad (4.6)$$

where we have used

$$\partial_k \tau = u_k.$$

The next term we have to evaluate is $\mathcal{M}[\varphi \perp f]$. Looking at the explicit form of $\mathcal{M}[\varphi]$ given in proposition 2.5.8 and observing that the integration occurs only in the hypersurfaces $s = \text{const.}$ (i.e. $\Sigma(s)$) we recognize that

$$\mathcal{M}[\varphi \perp f] = \mathcal{M}[\varphi] \perp f + \mathcal{M}[\varphi] \perp Df \quad (4.7)$$

An explicit expression for $\mathcal{M}[\varphi]$ is obtained as follows:

$$\begin{aligned} \sigma^b \sigma^c \nabla_{[b} (\varphi_{ac}) f \circ \tau &= \\ f \circ \tau \sigma^b \sigma^c \nabla_{[b} \varphi_{ac]} + \sigma^b \sigma^c \partial_{[b} \tau \varphi_{ac]} Df \circ \tau \end{aligned} \quad (4.8)$$

The two terms on the r.h.s. of (4.8) correspond to the two terms on the r.h.s. of (4.7). It is evident from the definitions that

$$\sigma^a(z, x) \partial_a \tau(x) = 0$$

for $\tau(x) = \tau(z) = s$. Therefore we get

$$\mu_a[\varphi] = - \int_0^1 dt (1-t) t^{-2} H_a^p \partial_p \tau \sigma^q \sigma^r \varphi_{qr}$$

with the same conventions as in section 2.5.

From this explicit expression it is clear that μ is a linear continuous map from $\mathcal{F}_{[2]}^0$ to $\mathcal{F}_1^0$ (compare propositions 2.5.9 - 2.5.12) and that $\mu(\mathcal{F}_{[2]}^0) \subset \mathcal{F}_1^0$.

In order to proceed further we observe that

$$\text{Exp}_Z^* (\psi \perp f) (X) = \Psi(z, X) f \circ \tau \circ \text{Exp}_Z X$$

where

$$\Psi(z, X) = \text{Exp}_Z^* \psi(X).$$

Therefore

$$\nabla_{*u} \text{Exp}_Z^* (\psi \perp f) = \nabla_{*u} \Psi f \circ \tau \circ \text{Exp}_Z + \Psi h Df \circ \tau \circ \text{Exp}_Z$$

where

$$h^{-1} := D_S(u_k \sigma^k).$$

For $X \in \text{Exp}_Z^{-1}(\Sigma(s))$ we obtain

$$\begin{aligned} \nabla_{*u} \text{Exp}_{Z(s)}^* (\psi \perp f) (X) &= \nabla_{*u} \Psi(z(s), X) f(s) + \\ &+ \Psi(z(s), X) h(z(s), X) Df(s) \end{aligned} \quad (4.9)$$

and correspondingly

$$\begin{aligned} \nabla_{*u}^2 \text{Exp}_{Z(s)} (\psi \perp f) (X) &= \nabla_{*u}^2 \Psi(z(s), X) f(s) + \\ &+ (2 \nabla_{*u} \Psi(z(s), X) h(z(s), X) + \\ &+ \Psi(z(s), X) \nabla_{*u} h(z(s), X)) Df(s) + \\ &+ \Psi(z(s), X) h^2(z(s), X) D^2 f(s) \end{aligned} \quad (4.10)$$

Using (4.7) and proposition 4.1 we find that

$$\begin{aligned}
& \langle \hat{T}_{z(s)}, \text{Exp}_{z(s)}^* (\varphi \perp f) + G \cdot \text{Exp}_{z(s)}^* \mathcal{M}[\varphi \perp f] \rangle = \\
& = \sum_{q=0}^2 \langle \nabla_{*u}^q \bar{Q}_{z(s)}^{(q)}, \text{Exp}_{z(s)}^* (\varphi \perp f) + G \cdot \text{Exp}_{z(s)}^* (\mathcal{M}[\varphi] \perp f) + \\
& + G \cdot \text{Exp}_{z(s)}^* (\mathcal{M}[\varphi] \perp Df) \rangle
\end{aligned}$$

In virtue of (4.9) and (4.10) this may be written as

$$\langle \hat{T}_{z(s)}, \text{Exp}_{z(s)}^* (\varphi \perp f) + G \cdot \text{Exp}_{z(s)}^* \mathcal{M}[\varphi \perp f] \rangle = \quad (4.11)$$

$$= \sum_{q=0}^3 [\Theta_s^{(q)}, \varphi] D^q f(s)$$

where

$$[\Theta_s^{(0)}, \varphi] := \langle \hat{T}_{z(s)}, \text{Exp}_{z(s)}^* \varphi + G \cdot \text{Exp}_{z(s)}^* \mathcal{M}[\varphi] \rangle \quad (4.12)$$

$$[\Theta_s^{(1)}, \varphi] := \langle \hat{T}_{z(s)}, G \cdot \text{Exp}_{z(s)}^* \mathcal{M}[\varphi] \rangle - \quad (4.13)$$

$$- \langle \bar{Q}_{z(s)}^{(1)}, h(\text{Exp}_{z(s)}^* \varphi + G \cdot \text{Exp}_{z(s)}^* \mathcal{M}[\varphi]) \rangle -$$

$$- \langle \bar{Q}_{z(s)}^{(2)}, (\nabla_{*u}^h)(\text{Exp}_{z(s)}^* \varphi + G \cdot \text{Exp}_{z(s)}^* \mathcal{M}[\varphi]) \rangle -$$

$$- \langle \nabla_{*u} \bar{Q}_{z(s)}^{(2)}, 2 h(\text{Exp}_{z(s)}^* \varphi + G \cdot \text{Exp}_{z(s)}^* \mathcal{M}[\varphi]) \rangle$$

$$[\Theta_s^{(2)}, \varphi] := \langle \bar{Q}_{z(s)}^{(2)}, h^2(\text{Exp}_{z(s)}^* \varphi + \quad (4.14)$$

$$+ G \cdot \text{Exp}_{z(s)}^* \mathcal{M}[\varphi]) \rangle - \langle \bar{Q}_{z(s)}^{(1)}, G \cdot \text{Exp}_{z(s)}^* \mathcal{M}[\varphi] \rangle -$$

$$- \langle \bar{Q}_{z(s)}^{(2)}, \nabla_{*u}^h G \cdot \text{Exp}_{z(s)}^* \mathcal{M}[\varphi] \rangle -$$

$$- \langle \nabla_{*u} \bar{Q}_{z(s)}^{(2)}, 2 h G \cdot \text{Exp}_{z(s)}^* \mathcal{M}[\varphi] \rangle$$

$$[\Theta_s^{(3)}, \varphi] := \langle \bar{Q}_{z(s)}^{(2)}, h^2 G \cdot \text{Exp}_{z(s)}^* \mathcal{M}[\varphi] \rangle \quad (4.15)$$

Since $\{\hat{T}_Z\}$ (hypothesis) and $\{\bar{Q}_Z^{(q)}\}$ (proposition 4.1) are composable and in virtue of the properties of $\mathcal{M}$ and $\mathcal{M}$ (linearity, continuity) it is easy to see from (4.12) - (4.15) that

$$(i) \quad s \mapsto [\Theta_s^{(q)}, \varphi] \text{ is of class } C^\infty \text{ for all } \varphi \in \mathcal{E}_{[2]}^0 \text{ and } q = 0 \dots 3$$

$$(ii) \quad \varphi_n \xrightarrow{0} 0 \text{ implies } [\Theta_s^{(q)}, \varphi_n] \rightarrow 0$$

for all $s \in \mathbb{R}$ and $q = 0 \dots 3$.

Putting (4.5), (4.6) and (4.11) together one gets, after partial integration,

$$\langle \int_{\Sigma} T_{\varphi} \mid f \rangle = \int T_{\varphi}(s) f(s) ds$$

where

$$T_{\varphi}(s) = [P^k \dot{z}^l - D_s(u_m S^{mk} \dot{z}^l)] \varphi_{kl}(z(s)) + \quad (4.16)$$

$$+ (\delta_n^k - \dot{z}^k u_n) S^{nl} \dot{z}^m \nabla_k \varphi_{lm}(z(s)) + \sum_{q=0}^3 (-1)^q D^q [\Theta_s^{(q)}, \varphi]$$

$s \mapsto T_{\varphi}(s)$ is obviously of class C^∞ since all the factors in the different terms of (4.16) are of class C^∞ . Furthermore let $\varphi_n \xrightarrow{\mathcal{E}} 0$. Then $\varphi_n(z(s)) \xrightarrow{\mathbb{R}} 0$ and $(\nabla \varphi_n)(z(s)) \xrightarrow{\mathbb{R}} 0$ for all $s \in \mathbb{R}$. Furthermore, as we have seen above, $[\Theta_s^{(q)}, \varphi_n] \xrightarrow{\mathbb{R}} 0$ for all $s \in \mathbb{R}$ and $q = 0, \dots, 3$. In virtue of corollary 1.2.14 (a consequence of the Banach-Steinhaus-theorem, see also proposition 2.3.13) it follows that $D^q [\Theta_s^{(q)}, \varphi_n] \xrightarrow{\mathbb{R}} 0$, and therefore $T_{\varphi_n}(s) \xrightarrow{\mathbb{R}} 0$ for all $s \in \mathbb{R}$. $\square$

Nota: In view of (4.12) we may write (4.16) in the following form:

$$\begin{aligned} \int_{\Sigma(s)} T \Phi &= [P^{k\dot{z}^1} - D_S(u_m S^{mk} \dot{z}^1)] \Phi_{kl} \\ &+ (\delta_n^k - \dot{z}^k u_n) S^{n1} \dot{z}^m \nabla_k \Phi_{lm} + \\ &+ \sum_{q=1}^3 (-1)^q D^q [\Theta_s^{(q)}, \Phi] \end{aligned}$$

In a center of mass description ($P = Mu$, $P \cdot S = 0$) we find

$$\begin{aligned} \int_{\Sigma(s)} T \Phi &= P^{k\dot{z}^1} \Phi_{kl}(z(s)) + S^{k1} \dot{z}^m \nabla_k \Phi_{lm} \\ &+ \langle \hat{T}_{z(s)}, \exp_{z(s)} \Phi + G \cdot \exp_{z(s)} \mu[\Phi] \rangle \\ &+ \sum_{q=1}^3 (-1)^q D^q [\Theta_s^{(q)}, \Phi] \end{aligned}$$

4. APPLICATIONS OF THE DEVELOPPED CALCULUS

In chapters 1,2 and 3 we have extended Dixon's theory to admissible energy-momentum distributions and have filled the gaps described in the introduction. In the following chapter we give a few applications of the extended calculus and inquire how an energy condition restricts the nature of a distribution in $(\mathcal{F}_{[2]}^{\circ})'$.

4.1 Simple Multipole Particles

We derive the form of energy-momentum distributions of particularly simple multipole structure: quadrupole and monopole particles.

Proposition 1.1: Let $l \equiv z(s)$ be a smooth timelike curve, u be a t.f.d. unit vector field along l and P , S and I be smooth tensor fields along l satisfying the usual conditions for momentum, angular momentum and quadrupole moment (with respect to $u!$) as listed at the end of appendix 3.2A. Let J be related to I by (3.2.A13), i.e.

$$J^{lmnr} = I[l[nm]r]$$

$$I^{m_1 m_2 k l} = \frac{4}{3} J^{(m_1 | k | m_2) l}$$

where antisymmetrization is applied separately to (l,m) and (n,r) . Let P , S and I (resp. J) satisfy the integral laws of motion (see [10])

$$D_S P^k = \frac{1}{2} \dot{z}^l S^{mn} R^k_{lmn} + \frac{1}{6} J^{lmnr} \nabla^k R_{lmnr}$$

$$D_S S^{kl} = 2P^{[k} \dot{z}^{l]} - \frac{4}{3} R^{[k}_{mn} J^{l]mnr}$$

Then the corresponding energy-momentum distribution is of the following form

$$\begin{aligned} \langle T, \varphi \rangle = \int ds \left[(P^{k\dot{z}1} + \frac{2}{3} R_{mij}^{kI1mij}) \varphi_{k1} + \right. \\ \left. + S^{mk\dot{z}1} \nabla_m \varphi_{k1} + \frac{1}{2} I^{m_1 m_2 k1} \nabla_{m_1 m_2} \varphi_{k1} \right] \end{aligned} \quad (1.1)$$

which may also be expressed as

$$\begin{aligned} T^{k1}(x) = \int ds \left[(P^{(k\dot{z}1)} + \frac{2}{3} R_{mij}^{(kI1)mij}) \delta(x-z(s)) - \right. \\ \left. - S^{m(k\dot{z}1)} \nabla_m \delta(x-z(s)) + \frac{1}{2} I^{m_1 m_2 k1} \nabla_{m_1 m_2} \delta(x-z(s)) \right] \end{aligned}$$

Proof: According to theorem 3.4.2 and the definition of the moments (def. 3.2A.3) we find that

$$\begin{aligned} \langle T, \varphi \rangle = \int ds \left(P^{k\dot{z}1} \varphi_{k1} + S^{mk\dot{z}1} \nabla_m \varphi_{k1} + \right. \\ \left. + \frac{1}{2} I^{m_1 m_2 k1} \langle \nabla_{*m_1 m_2} \delta, (\text{Exp}_{z(s)}^* \varphi)_{k1} + \right. \\ \left. + (G \cdot \text{Exp}_{z(s)}^* \omega[\varphi])_{k1} \rangle \right) = \\ = \int ds \left(P^{k\dot{z}1} \varphi_{k1} + S^{mk\dot{z}1} \nabla_m \varphi_{k1} + \right. \\ \left. + \frac{1}{2} I^{m_1 m_2 k1} \langle \nabla_{m_1 m_2} H_k^a H_l^b \varphi_{ab} \rangle + \right. \\ \left. + \frac{1}{2} I^{m_1 m_2 k1} \langle \nabla_{*m_1 m_2} G_{kl}^n H_n^a \omega_a[\varphi] \rangle \right) \end{aligned} \quad (1.2)$$

It remains to evaluate the coincidence limits:

$$\begin{aligned}
\langle \nabla_{*(m_1 m_2)}^{H(k^a H_1)^b} \Phi_{ab} \rangle &= \langle \nabla_{*(m_1 m_2)}^{H(k^a H_1)^b} \rangle \Phi_{ab}(z) + \\
&+ 2 \langle \nabla_{*(m_1 l)}^{H(k^a H_1)^b} \rangle \delta_{lm_2}^c \nabla_c \Phi_{ab}(z) + \\
&+ \langle \nabla_{*(m_1 H_{m_2})}^c \nabla_c \Phi_{ab} \rangle \delta_k^a \delta_l^b + \nabla_{(m_1 m_2)} \Phi_{kl}
\end{aligned}$$

It follows from (2.2.A26) and (2.2.A28) that

$$\begin{aligned}
\langle \nabla_{*(m_1 m_2)}^{H(k^a H_1)^b} \Phi_{ab} \rangle &= \\
&= \nabla_{(m_1 m_2)} \Phi_{kl} + \frac{4}{3} R^n_{(m_1 m_2)} (k \Phi_l)_n
\end{aligned} \tag{1.3}$$

The other coincidence limit to be evaluated is

$$\begin{aligned}
\langle \nabla_{*(m_1 m_2)}^{G_{kl}^n H_n^a} \mathcal{M}_a[\Phi] \rangle &= \\
&= \langle \nabla_{*(m_1 m_2)}^{G_{kl}^n} \rangle \langle H_n^a \mathcal{M}_a[\Phi] \rangle + \\
&+ 2 \langle \nabla_{*(m_1 l)}^{G_{kl}^n} \rangle \langle \nabla_{*(l m_2)}^{H_n^a} \mathcal{M}_a[\Phi] \rangle + \\
&+ \langle G_{kl}^n \rangle \langle \nabla_{*(m_1 m_2)}^{H_n^a} \mathcal{M}_a[\Phi] \rangle
\end{aligned}$$

The first and third terms on the right hand side vanish in virtue of (2.2A.33) and (2.5.12a). Furthermore in view of (2.2A.26) and (2.5.12b)

$$\langle \nabla_{*m}^{H_n^a} \mathcal{M}_a[\Phi] \rangle = \Phi_{mn}(z)$$

Thus it follows from (2.2.A34)

$$\langle \nabla_{* (m_1 m_2)}^{G_{kl}^n H_n^a} \mathcal{M}_a[\Phi] \rangle = \frac{4}{3} R^n_{(kl) (m_1 \Phi_{m_2}) n} \quad (1.4)$$

Finally we observe that

$$\begin{aligned} & 4 (R^n_{(kl) (m_1 \Phi_{m_2}) n} - R^n_{(m_1 m_2) (k \Phi_l) n}) I^{m_1 m_2 kl} = \\ & = (R^n_{kl m_1} \Phi_{m_2 n} + R^n_{kl m_2} \Phi_{m_1 n} + R^n_{l k m_1} \Phi_{m_2 n} + \\ & + R^n_{l k m_2} \Phi_{m_1 n} - R^n_{m_1 m_2 k} \Phi_{ln} - R^n_{m_1 m_2 l} \Phi_{kn} - \\ & - R^n_{m_2 m_1 k} \Phi_{ln} - R^n_{m_2 m_1 l} \Phi_{kn}) I^{m_1 m_2 kl} = \\ & = - I^{m_1 m_2 kl} (R^n_{km_1 l} \Phi_{m_2 n} + R^n_{km_2 l} \Phi_{m_1 n} + \\ & + R^n_{lm_1 k} \Phi_{m_2 n} + R^n_{lm_2 k} \Phi_{m_1 n} + R^n_{m_1 m_2 k} \Phi_{ln} + \\ & + R^n_{m_2 m_1 k} \Phi_{ln} + R^n_{m_1 m_2 l} \Phi_{kn} + R^n_{m_2 m_1 l} \Phi_{kn} + \\ & + R^n_{m_1 k l} \Phi_{m_2 n} + R^n_{m_1 l k} \Phi_{m_2 n} + R^n_{m_2 k l} \Phi_{m_1 n} + \\ & + R^n_{m_2 l k} \Phi_{m_1 n}) = \\ & = - 6 I^{m_1 m_2 kl} (R^n_{(m_1 m_2 | l | \Phi_k) n} + R^n_{(m_1 m_2 | k | \Phi_l) n}) \end{aligned}$$

Both terms vanish in virtue of equation (3.2A.12). Therefore

$$(R^n_{kl m_1} \Phi_{m_2 n} - R^n_{m_1 m_2 k} \Phi_{ln}) I^{m_1 m_2 kl} = 0 \quad (1.5)$$

We insert (1.3) and (1.4) in (1.2), use (1.5) and get (1.1). $\square$

def. 5.1: We call a representation of a $T \in \mathcal{M}_\Sigma$ by $(l, u; P, S, \hat{T})$ a centre-of-mass description (see [10, 13, 20]) iff $P = Mu$ and $P \cdot S = 0$.

Proposition 1.2: An energy-momentum distribution such that all moments but the monopole vanish has to be of the following form

$$\langle T, \phi \rangle = \int ds \, M \dot{z}^k \dot{z}^l \phi_{kl}(z(s))$$

or

$$T^{kl} = \int ds \, M \dot{z}^k \dot{z}^l \delta(s - z(s))$$

The integral laws of motion imply $l \equiv \text{supp}(T)$ to be a geodesic. If we parametrize l by $g(\dot{z}, \dot{z}) = 1$ instead of $g(\dot{z}, u) = 1$ we find M to be a constant. This in turn shows that the choice of u only determines the parametrization of l . Thus if we choose $u = \dot{z}$ the description becomes a centre-of-mass description.

Proof: By hypothesis $S = 0$, $\hat{T} = 0$, and therefore $F = 0$ and $L = 0$. We get from theorem 3.4.2

$$\langle T, \phi \rangle = \int ds \, p^k \dot{z}^l \phi_{kl}$$

The equations of motion become

$$D_S P = 0 \tag{1.6a}$$

$$P \wedge \dot{z} = 0 \tag{1.6b}$$

Transvection of (1.6b) with u yields

$$P = \mu \dot{z} \tag{1.7}$$

$P \neq 0$ implies $\mu \neq 0$. From (1.7) follows that P is timelike. We define

$$M := g(P, P)^{1/2}$$

(1.6a) implies that

$$D_S M = 0$$

Let $s := f(t)$ be a reparametrization of l given by

$$D_t f = M^{-1} \mu \circ f$$

The tangent vector $v := \partial_t(z \circ f)$ satisfies

$$v = M^{-1} \mu \dot{z}$$

Whence, using (1.6a),

$$D_t v = M^{-2} \mu D_S (\mu \dot{z}) = 0$$

This shows that l is a geodesic and t is an affine parameter such that

$$g(v, v) = M^{-2} g(P, P) = 1$$

Finally we get

$$\langle T, \phi \rangle = \int dt M^{-1} \mu^2 \dot{z}^k \dot{z}^l \phi_{kl} = M \int v^k v^l \phi_{kl} dt \quad \square$$

4.2 The Dominant Energy Condition

In this section we derive some consequences of a generalization of the dominant energy condition of Hawking and Ellis [18].

def. 2.1: Let $(\mathcal{F}_S^r)^*$ denote the algebraic dual of $\mathcal{F}_S^r$, i.e. the space of all linear functionals $L : \mathcal{F}_S^r \rightarrow \mathbb{R}$. $L \in (\mathcal{F}_{[2]}^o)^*$ is said to satisfy the dominant energy condition iff for all non-spacelike future directed $\psi, \omega \in \mathcal{F}_1^o$

$$L[\psi \otimes \omega] \geq 0$$

Proposition 2.1: Let $L \in (\mathcal{T}_{[2]}^0)^*$ be a symmetric continuous tensor field. Then the dominant energy condition of definition 2.1 is equivalent to the dominant energy condition defined by Hawking and Ellis [18], i.e. for all $x \in M$ and all timelike $W \in M_x^*$, $L^{ab}(x) W_a W_b \geq 0$ and $L^{ab}(x) W_a$ is a non-spacelike vector.

Proof: (1): def. 2.1 implies that for all non-negative functions f and all non-spacelike future directed $\psi, \omega \in \mathcal{T}_1^0$

$$\int f(x) L^{ab}(x) \psi_a(x) \omega_b(x) \eta(x) \geq 0$$

Since L is continuous, it follows that for all $x \in M$ and all non-spacelike future directed $X, Y \in M_x^*$

$$L^{ab}(x) X_a Y_b \geq 0 \quad (2.1)$$

Let $W \in M_x^*$ be timelike. Put

$$i(W) := \begin{cases} +1 & \text{if } W \text{ is future directed} \\ -1 & \text{if } W \text{ is past directed} \end{cases}$$

Then $i(W)W$ is t.f.d. (2.1) implies that

$$L^{ab}(x) W_a W_b = L^{ab}(x) i(W) W_a i(W) W_b \geq 0 \quad (2.2)$$

and that

$$i(W) L^{ab} W_a X_b \geq 0$$

for all future directed non-spacelike $X \in M_x^*$. Thus $L^{ab}(x) W_a$ is non-spacelike. This, together with (2.2) gives the dominant energy condition of [18].

(2) It follows from the continuity of the map $W_a \mapsto L^{ab}(x) W_b$ and the dominant energy condition of [18] that for all x and all non-spacelike $W \in M_x^*$, $L^{ab}(x) W_a W_b \geq 0$ and $L^{ab}(x) W_b$ is a non-spacelike vector. Thus $L^{ab}(x) W_b$ is future directed for all non-spacelike W . This in turn implies that $L^{ab}(x) X_a Y_b \geq 0$ for all non-spacelike future directed $X, Y \in M_x^*$. $\square$

Proposition 2.2: If $L \in (\mathcal{T}_{[2]}^0)^*$ satisfies the dominant energy condition then $L \in (\mathcal{T}_{[2]}^0)'$ and L is of order 0.

Proof: Let ω^a , $a = 1..4$, be an orthonormal frame field such that

$$g = - \sum_{\alpha=1}^3 \omega^\alpha \otimes \omega^\alpha + \omega^4 \otimes \omega^4 \quad (2.3a)$$

and

$$e = \sum_{a=1}^4 \omega^a \otimes \omega^a \quad (2.3b)$$

Define

$$t^a := \omega^4 + \frac{1}{2} \omega^a$$

The $t^a(x)$, $a = 1..4$, are t.f.d. and linear independent for all $x \in M$:

$$g(t^a, t^a) = g(\omega^4, \omega^4) + g(\omega^4, \omega^a) + \frac{1}{4} g(\omega^a, \omega^a)$$

Therefore

$$g(t^\alpha, t^\alpha) = \frac{3}{4} \quad (\alpha = 1..3)$$

$$g(t^4, t^4) = \frac{9}{4}$$

Furthermore

$$g(t^\alpha, \omega^4) = 1 \quad (\alpha = 1..3)$$

$$g(t^4, \omega^4) = \frac{3}{2}$$

Finally

$$\omega^a = 2(t^a - \frac{2}{3} t^4) \quad (2.4)$$

From (2.3b) and (2.4) it is obvious that there are constants C_s^r such that for all $x \in M$ and all $\varphi \in T_s^r M$

$$\max_{a_1 \dots a_r, b_1 \dots b_s} \left| \varphi_{b_1 \dots b_s}^{a_1 \dots a_r}(x) \right| \leq C_s^r \left| \varphi(x) \right|_e$$

where $\varphi_{b_1 \dots b_s}^{a_1 \dots a_r}$ denotes the components of φ with respect to the base $\{t^a \mid a=1..4\}$. From the dominant energy condition one knows that the linear functional

$$\mathcal{F} \ni f \mapsto L[f t^a \otimes t^b]$$

is positive. Thus it is a positive measure (see, for example, [23]) and, a fortiori a distribution of order 0. Therefore, for all $f \in \mathcal{F}$, there exists a $v \in \mathcal{F}^+$ and a $C^{ab} > 0$ such that for all $f \in \mathcal{F}(F)$

$$\left| L[f t^a \otimes t^b] \right| \leq C^{ab} \sup_{x \in F \cap V} |f(x)|_e \quad (2.5)$$

Hence

$$\begin{aligned} |L[\varphi]| &\leq \sum_{a,b} |L[\varphi_{ab} t^a \otimes t^b]| \leq \\ &\leq \sum_{a,b} C^{ab} \sup_{x \in F \cap V} |\varphi_{ab}(x)| \leq C \sup_{x \in F \cap V} |\varphi(x)|_e \end{aligned} \quad (2.6)$$

where C is given by

$$c := c_2^0 \sum_{a,b} c^{ab}$$

(2.6) may be written in the following form: For all $F \in \mathcal{F}$ there exist a $c > 0$ and a $v \in \mathcal{F}$ such that for all $\varphi \in \mathcal{T}_{[2]}^0(F)$

$$|L[\varphi]| \leq c p_{V,0}(\varphi)$$

Proposition 1.4.2 and definition 1.4.2 complete the proof. $\square$

Remark: The dominant energy condition is not sufficient to prove that the momentum is a non-spacelike vector field along l for general T (for a point particle it will turn out to be sufficient: See theorem 2.6 and proposition 1.2). In order to get a timelike momentum, one has to impose a stronger energy condition (this is even true for the case of a smooth energy momentum tensor, confer [20, 21]): Let $T \in \mathcal{M}_\Sigma$ satisfy the dominant energy condition. Then (2.5) implies that there exist positive measures T^{ab} such that

$$\langle T, f t^a \otimes t^b \rangle = \langle T^{ab}, f \rangle$$

This in turn implies (see [5], (13.13.1) and (13.15.2) that there exist a positive measure μ and locally integrable functions T^{ab} such that

$$\langle T, f t^a \otimes t^b \rangle = \langle \mu, T^{ab} f \rangle$$

Using the theorem of Riesz-Markov (see, e.g. [2], p. 47) this implies that there exist a regular Borel measure m and a locally integrable tensor field T such that

$$\langle T, \varphi \rangle = \int T \cdot \varphi \, dm$$

Now we may impose on T the additional condition that $T(x)$

maps almost everywhere the interior of the future light cone into itself (see condition 2.2(a) in [20]). Reasoning in a way similar to [20, 21] we obtain, that P is timelike provided the curvature is not too large.

Lemma 2.3: Let $T \in (\mathcal{F}_S^r)'$ be of order 0. Suppose that $\text{supp}(T) = l \equiv z(s)$ is a smooth timelike curve contained in W . Furthermore let $\{\Sigma(s) \mid s \in \mathbb{R}\}$ and $\{\Pi(t) \mid t \in \mathbb{R}\}$ be two spacelike C^∞ foliations of W . The corresponding time-functions are τ and π , i.e.

$$x \in \Sigma(s) \quad \Leftrightarrow: \quad \tau(x) = s$$

$$y \in \Pi(t) \quad \Leftrightarrow: \quad \pi(y) = t$$

Let (Σ, τ) and (Π, π) satisfy the conditions of section 2.3. Then there exists a diffeomorphism $q: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\int_{\Pi} T \Phi = q \int_{\Sigma} T \Phi$$

Proof: Since $\{\Sigma\}$ and $\{\Pi\}$ are smooth spacelike foliations and l is a smooth timelike curve the following defines a diffeomorphism $q: \mathbb{R} \rightarrow \mathbb{R}$

$$q(s) = t: \quad \Leftrightarrow \quad \Sigma(s) \cap l = \Pi(t) \cap l$$

This implies that

$$(\pi|_l) = (q \circ \tau|_l)$$

Since T is of order 0, $\langle T, \Phi \rangle$ depends only upon $(\Phi|_l)$. Therefore

$$\begin{aligned} \left\langle \int_{\Pi} T \Phi \mid f \right\rangle &= \langle T, \Phi \circ f \circ \pi \rangle = \\ &= \langle T, \Phi \circ f \circ q \circ \tau \rangle = \left\langle \int_{\Sigma} T \Phi \mid f \circ q \right\rangle \quad \square \end{aligned}$$

Corollary 2.4: Under the hypotheses of Lemma 2.3 exists a parametrization of $\{n\}$ such that for all $s \in \mathbb{R}$

$$\Sigma(s) \cap 1 = \Pi(s) \cap 1$$

In this case we get

$$\int_{\Sigma} T \Phi = \int_{\Pi} T \Phi \quad \square$$

Lemma 2.5: Let $T \in (\mathcal{F}_{[2]}^0)'$ be a distribution of order 0 satisfying the dominant energy condition and the local law of motion $\nabla \cdot T = 0$. Let furthermore $\text{supp}(T)$ be a smooth timelike curve $l \equiv z(t)$. Then there exists a smooth symmetric $\binom{2}{0}$ tensor field T along l such that

$$\langle T, \Phi \rangle = \int T(t) \Phi(z(t)) dt$$

Proof: T is of order 0, i.e. for all $F \in \mathcal{F}$ there exist a $V \in \mathcal{F}^+$ and a $C > 0$ such that for all $\Phi \in \mathcal{F}_{[2]}^0(F)$

$$|\langle T, \Phi \rangle| \leq C \sup_{x \in V \cap F} |\Phi(x)|_e$$

Therefore there exist for all $\Phi \in \mathcal{F}_{[2]}^0$ a $K(\Phi) > 0$ and a compact $I \subset \mathbb{R}$ such that

$$|\langle \int_{\Sigma} T \Phi | f \rangle| \leq K(\Phi) \sup_{t \in I} |f(t)|$$

and

$$I \supset \tau(\text{supp}(T) \cap \text{supp}(\Phi))$$

This implies that for all $\Phi \in \mathcal{F}_{[2]}^0$, $\int_{\Sigma} T \Phi$ is a measure of compact support. Using Lemma 2.3.11 and the local law of motion we deduce that for all $\lambda \in \mathcal{F}_1^0$

$$\int_{\Sigma} T^{\alpha\beta} \lambda = -D \int_{\Sigma} T^{\alpha\beta} \partial \tau \otimes \lambda$$

and, therefore

$$\int_{\Sigma} T^{\alpha\beta} \partial \tau \otimes \lambda = - \int_{-\infty}^{\infty} \int_{\Sigma} T^{\alpha\beta} \lambda \quad (2.7)$$

This implies that for all $\lambda \in \mathcal{F}_1^0$, $\int_{\Sigma} T^{\alpha\beta} \partial \tau \otimes \lambda$ is a Lebesgue integrable function of bounded variation and compact support. From the dominant energy condition it follows that $\int_{\Sigma} T^{\alpha\beta} \partial \tau \otimes \lambda$ is non-negative for all non-spacelike future directed $\lambda \in \mathcal{F}_1^0$. Let ω^a be an orthonormal frame field satisfying $\omega^4 = \chi \partial \tau$ with $\chi > 0$, then we define for $\alpha \in \mathcal{F}$

$$T^{ab}[\alpha] := \int_{\Sigma} T^{\alpha\beta} \omega^a \otimes \omega^b$$

The $T^{ab}[\alpha]$ are integrable functions of compact support and bounded variation, and $T^{44}[\alpha] \geq 0$ for $\alpha \geq 0$. Let $\beta, \gamma, \nu, \mu \in \mathcal{F}$, $|\nu| \leq \beta$, $|\mu| \leq \gamma$ and $M, N \in \{0, 1\}$. We define

$$\begin{aligned} X &:= \beta \omega^4 + \nu (-1)^N \omega^a \\ Y &:= \gamma \omega^4 + \mu (-1)^M \omega^a \end{aligned}$$

Using the energy condition we find

$$\begin{aligned} &(-1)^{M+N} T^{ab}[\mu\nu] + (-1)^N T^{a4}[\gamma\nu] + \\ &+ (-1)^M T^{4b}[\beta\mu] + T^{44}[\beta\gamma] \geq 0 \end{aligned} \quad (2.8)$$

From (2.8) we deduce the following inequalities

$$M = 0, N = 0 : -T^{ab}[\mu\nu] - T^{a4}[\gamma\nu] - T^{4b}[\mu\beta] \leq T^{44}[\beta\gamma] \quad (2.9)$$

$$M = 1, N = 1 : -T^{ab}[\mu\nu] + T^{a4}[\gamma\nu] + T^{4b}[\mu\beta] \leq T^{44}[\beta\gamma] \quad (2.10)$$

Adding (2.9) and (2.10) we find

$$-T^{ab}[\mu\nu] \leq T^{44}[\beta\gamma]$$

Analogously we deduce from the cases $M = 1, N = 0$ and $M = 0, N = 1$

$$T^{ab}[\mu\nu] \leq T^{44}[\beta\gamma]$$

Defining $\kappa := \mu\nu$ and $\alpha := \beta\gamma$ and observing that every $\alpha \in \mathcal{F}$ may be written as $\alpha = \beta\gamma$, $\beta \in \mathcal{F}$, $\gamma \in \mathcal{F}$, we deduce that

$$|T^{ab}[\kappa]| \leq T^{44}[\alpha]$$

whenever $|\kappa| \leq \alpha$. It is trivial that for every $\kappa \in \mathcal{F}$ there exists a $\alpha \in \mathcal{F}$ such that $|\kappa| \leq \alpha$. According to a wellknown theorem of measure theory this implies the existence of locally integrable (with respect to the measure $T^{44}[\alpha]!$) functions $h_{\kappa, \alpha}^{ab}$ such that

$$T^{ab}[\kappa] = h_{\kappa, \alpha}^{ab} T^{44}[\alpha]$$

But $T^{44}[\alpha]$ is a Lebesgue integrable function, say t_α , i.e.

$$\langle T^{44}[\alpha] | f \rangle = \int t_\alpha(s) f(s) ds$$

This implies that $h_{\kappa, \alpha}^{ab} t_\alpha$ is a locally integrable function (see, for example [5], (13.14.5)), i.e. that for all $\kappa \in \mathcal{F}$, $T^{ab}[\kappa]$ is a locally integrable function.

Now let l be parametrized by $z(s)$ such that $\tau \circ z = \text{id}_{\mathbb{R}}$. Then $(\alpha|1) = (\alpha \circ z \circ \tau | 1)$ for all $\alpha \in \mathcal{F}$. Furthermore there exists a smooth partition of unity $\{\chi_n : \mathbb{R} \rightarrow [0, 1] \mid n \in \mathbb{Z}\}$ such that $\text{supp}(\chi_n) = [n-1, n+1]$. Then $\chi_n \circ \tau \in \mathcal{F}$ and for every $F \in \mathcal{F}$ there exist two integers $N < M \in \mathbb{Z}$ such that

$$\left(\sum_{n=N}^M \chi_n \mid F \cap \text{supp}(T) \right) = 1$$

For all $n \in \mathbb{Z}$ and for all $a, b = 1, \dots, 4$ we define locally integrable functions T_n^{ab} by

$$T_n^{ab} := T^{ab}[\chi_n \circ \tau]$$

Since $\text{supp}(T_n^{ab}) \subset [n-1, n+1]$, the sums

$$T^{ab} := \sum_n T_n^{ab}$$

are locally finite and define therefore locally integrable functions.

Since T is of order 0, $\langle T, \varphi \rangle$ depends only upon $(\varphi|_1)$. Therefore

$$\begin{aligned} \langle T, \varphi \rangle &= \langle T, \sum_{n=N}^M \chi_n \circ \tau \sum_{a,b} \omega^a \otimes \omega^b \varphi_{ab} \circ z \circ \tau \rangle \\ &= \sum_{n=N}^M \sum_{a,b} \int T_n^{ab}(s) \varphi_{ab}(z(s)) ds \end{aligned}$$

Since for $n < N$ and $n > M$

$$\int T_n^{ab}(s) \varphi_{ab}(z(s)) ds = 0$$

we get finally

$$\langle T, \varphi \rangle = \sum_{a,b} \int T^{ab}(s) \varphi_{ab}(z(s)) ds$$

whence it follows immediately that

$$\int_{\Sigma} T \varphi = \sum_{a,b} T^{ab} \varphi_{ab} \circ z \quad (2.11)$$

Since the left hand side of (2.11) is an invariant, the same is true for the right hand side. This shows that $T^{ab}(s)$ may be considered as components of a symmetric tensor field along l with respect to the orthonormal base dual to $\{\omega^a\}_{a=1\dots 4}$

So far we have shown that there exists a locally integrable symmetric tensor field T along l such that

$$\langle T, \varphi \rangle = \int T(s) \cdot \varphi(z(s)) ds$$

and

$$\int_{\Sigma(s)} T \varphi = T(s) \cdot \varphi(z(s)) \quad (2.12)$$

Using (2.7) again, we see that for all $\lambda \in \mathcal{F}_1^0$ the function $s \mapsto \mathcal{P}^{ab}(a) \partial_a \tau(z(s)) \lambda_b(z(s))$ is uniformly continuous and of compact support. For all vector fields $\mathcal{J}$ there exist positive functions $\alpha, \varepsilon: M \rightarrow \mathbb{R}$ and a spacelike foliation $\{\pi_{\mathcal{J}}(s)\}$ of a tube containing l , whose time function shall be denoted by $\pi_{\mathcal{J}}^{-1}$, such that $(\partial \pi_{\mathcal{J}} | l) = (\alpha \partial \tau + \alpha \varepsilon \mathcal{J} | l)$ and $(\pi_{\mathcal{J}} | l) = (\tau | l)$: We parametrize l by $\{z(s)\} = \Sigma(s) \cap l$ and define $\pi_{\mathcal{J}}(s)$ to be the hypersurface consisting of all geodesics orthogonal to $\partial \tau(z(s)) + \varepsilon(z(s)) \mathcal{J}(z(s))$; if we choose ε to be small enough this will be a t.f.d. vector field defining a smooth spacelike foliation in a neighbourhood of l . α is determined by the chosen parametrization of $\pi_{\mathcal{J}}$. We get from corollary 2.4

$$\int_{\pi_{\mathcal{J}}} T \cdot \Phi = \int_{\Sigma} T \cdot \Phi$$

and therefore

$$\begin{aligned} \int_{\Sigma(s)} T \cdot (\partial \tau + \varepsilon \mathcal{J}) \otimes \frac{1}{\varepsilon} \lambda &= \int_{\pi_{\mathcal{J}}(s)} T \cdot \partial \pi_{\mathcal{J}} \otimes \frac{1}{\alpha \varepsilon} \lambda - \\ &= - \int_{-\infty}^s ds \int_{\pi_{\mathcal{J}}(s)} T \cdot \mathcal{J} \otimes \left(\frac{1}{\alpha \varepsilon} \lambda \right) = - \int_{-\infty}^s \int_{\Sigma(s)} T \cdot \mathcal{J} \otimes \left(\frac{1}{\alpha \varepsilon} \lambda \right) \end{aligned}$$

or

$$\begin{aligned} \int_{\Sigma(s)} T \cdot \mathcal{J} \otimes \lambda &= - \int_{-\infty}^s ds \int_{\Sigma(s)} T \cdot \mathcal{J} \otimes \left(\frac{1}{\alpha \varepsilon} \lambda \right) - \\ &\quad - \int_{\Sigma(s)} T \cdot \partial \tau \otimes \frac{\lambda}{\varepsilon} \end{aligned} \quad (2.13)$$

¹⁾ i.e. $\pi_{\mathcal{J}}(x) = s : \Leftrightarrow x \in \pi_{\mathcal{J}}(s)$

Both terms on the right hand side are uniformly continuous. We deduce that for all ζ and $\lambda \in \mathcal{F}_1^0$, $\int_{\Sigma} T \zeta \otimes \lambda$ is uniformly continuous which implies the uniform continuity of $\int_{\Sigma} T \phi$ for all $\phi \in \mathcal{F}_{[2]}^0$. Using equations (2.7) and (2.13) and the above arguments again and again, we find that $\int_{\Sigma} T \phi$ is of class C^∞ for all $\phi \in \mathcal{F}_{[2]}^0$. In view of equation (2.12) this yields the smoothness of T . $\square$

Theorem 2.6: ("A point particle has no structure") The motion of a point-particle may be described by a linear functional $T \in (\mathcal{F}_{[2]}^0)^*$ (not a priori a distribution!) satisfying

- (i) The dominant energy condition
- (ii) the "law of motion": for all $\omega \in \mathcal{F}_1^0$, $T[\omega] = 0$
- (iii) the "point" condition: There exists a smooth timelike curve l such that $T[\phi] = 0$ for all $\phi \in \mathcal{F}_{[2]}^0$ satisfying $\text{supp}(\phi) \subset M \setminus l$.

Then l is a geodesic, and

$$T[\phi] = M \int \dot{z}^k(s) \dot{z}^l(s) \phi_{kl}(z(s)) ds$$

where M is a positive constant and $s \mapsto z(s)$ is an affine parametrization of l .

Proof: From proposition 2.2 it follows that T is a distribution of order 0. It obviously satisfies the hypotheses of Lemma 5.7. Therefore

$$T[\phi] = \int T(s) \cdot \phi(z(s)) ds$$

where $s \mapsto z(s)$ is a smooth parametrization of l . Choosing any smooth timelike unit vector field u along l and defining $\Sigma(s) = \Sigma(z(s), u(s))$ we see immediately that $T \in \mathcal{M}_{\Sigma}$. Using the definitions we find for momentum and angular momentum

$$p^k(s) = T^{k1}(s) u_1(s)$$

$$s^{k1}(s) = 2 \langle H^{[k}{}_a \sigma^{1]} T^{am} u_m \rangle = 0$$

Furthermore for every $\Psi \in (T_{[2]}^0, T)$, $\text{Exp}_{z(s)}^{*-1} \mathcal{P}[\Psi] \in \mathcal{H}$. But $\beta \in \mathcal{H}$ implies $\langle \beta \rangle = 0$ (see proposition 2.5.1). In addition it follows from (2.5.25) that $\langle \mu[\beta](z, x) \rangle = 0$. This yields

$$\hat{T}_{z(s)} = 0$$

Proposition 1.2 completes the proof. $\square$

Remark: This generalizes results of Tulczyjew [29] and Havas and Goldberg [17]. Comparison of the proofs of [17, 29] with the one of the above theorem shows that use of Dixon's theory allows a substantial simplification.

Theorem 2.7: For a body (particle) described by an admissible energy momentum distribution satisfying the dominant energy condition there are only two possibilities:

- (i) it is a simple monopole particle
- (ii) it has non-zero multipole moments of arbitrary high order

Proof: Suppose there was a $T \in \mathcal{M}_\Sigma$ satisfying the dominant energy condition such that all but finitely many multipole moments vanish. Then it follows from theorem 3.4.2 that $\text{supp}(T) = 1$. This shows that the hypotheses of theorem 2.6 are satisfied. This implies that T describes a simple monopole particle. $\square$

Remark: For flat space-time a more quantitative result concerning the minimal radius r of a convex body with mass M and spin tensor S has been found by Møller [19], namely

$$r^2 \geq \frac{1}{2} \frac{s^{kl} s_{kl}}{M^2}$$

For a derivation of this result see, for example [25] pp. 314-315.

ACKNOWLEDGEMENTS

I am very grateful to Prof. J. Ehlers, who supervised this work. His critical comments and many helpful discussions with him - especially after his reading of the manuscript - clarified, improved or extended the work.

Stimulating impulses came from the members of the relativity group at the Max-Planck-Institut für Physik und Astrophysik, especially from Drs. D. Christodoulou, B. G. Schmidt and M. Streubel, for which I am grateful.

Furthermore I wish to thank Drs. W. G. Dixon and W. M. Tulczyjew for their interest and suggestions.

Many thanks are also due to Mrs. U. Hennig for her patience and dilligence in typing the manuscript.

Finally I wish to thank Prof. H.J. Meister for clarifying certain points of the manuscript.

5.1 List of "Properties" Occurring in Certain Definitions

property	definition	page
(CF 1,2)	2.3.3	53
(E 1-4)	3.1.4	87
(K 1,2)	1.1.1	12
(K 3)	1.1.2	12
(K 2', 3')	1.1.3	13
(SC 1,2)	2.3.4	56
(SN 1,2)	1.2.2	17
(T 1-3)	1.3.1	20
(TVS 1-3)	1.2.1	17

5.2 BIBLIOGRAPHY

- [1] Cartan, H. (1967): Calcul différentiel, Herman, Paris
- [2] Choquet-Bruhat, Y., DeWitt-Morette, C. and Dillard-Bleick, M. (1977): Analysis, Manifolds and Physics, North-Holland, Amsterdam
- [3] Cristescu, R. (1964): Rev. Rouman. Math. Pures et Appl., IX, 8, 703
- [4] DeWitt, B.S., and Brehme, R.W. (1960): Ann. Phys. (N.Y.), 9, 220
- [5] Dieudonné, J. (1974): Eléments d'analyse, tome 2, Gauthier-Villars, Paris
- [6] Dixon, W.G. (1970): Proc. R. Soc. London, A314, 499
- [7] Dixon, W.G. (1970): Proc. R. Soc. London, A319, 509
- [8] Dixon, W.G. (1973): Gen. Rel. Grav., 4, 199
- [9] Dixon, W.G. (1974): Trans. R. Soc. London, A277, 59
- [10] Dixon, W.G. (1979): pp. 156-219 in [15]
- [11] Edwards, R.E. (1965): Functional Analysis, Holt, Rinehart and Winston, New York
- [12] Ehlers, J., Rosenblum, A., Goldberg., J.N. and Havas, P. (1976): Ap. J., 208, L77
- [13] Ehlers, J., and Rudolph, E. (1977): Gen. Rel. Grav., 8, 197
- [14] Ehlers, J. (1977): pp. 45-73 in Proceedings of the Int. School of General Relativistic Effects in Physics and Astrophysics: Experiments and Theory (3rd Course), J. Ehlers (ed.), MPI-PAE/Astro 138, Munich
- [15] Ehlers, J. (Ed.) (1979): Course 67 of the Int. School of Physics "Enrico Fermi", North-Holland, Amsterdam
- [16] Esposito, F.P., and Witten, L. (Eds) (1977): Asymptotic Structure of Spacetime, Plenum, New York
- [17] Havas, P., Goldberg, J.N. (1962): Phys. Rev., 128, 398

- [18] Hawking, S.W., and Ellis, G.F.R. (1973): The Large Scale Structure of Space-Time, Cambridge University Press, Cambridge
- [19] Møller, C. (1949): Commun. Dublin Inst. Adv. Stud. A5, 1
- [20] Schattner, R. (1979): Gen. Rel. Grav., 10, 377 and 395
- [21] Schattner, R. (1977): The Centre of Mass in General Relativity (Diplomarbeit) MPI-PAE/Astro 122, Munich
- [22] Schwartz, L. (Notes by P. Rabinowitz, 1964): Functional Analysis Courant Institute of Mathematical Sciences, New York University
- [23] Schwartz, L. (1973): Théorie des distributions, Herman, Paris
- [24] Streubel, M. (1978): 'Conserved' Quantities Related to Asymptotic Symmetries for Isolated Systems in General Relativity (Dissertation), MPI-PAE/Astro 165, Munich
- [25] Synge, J.L. (1956): Relativity: The Special Theory, North-Holland, Amsterdam
- [26] Synge, J.L. (1971): Relativity: The General Theory, North-Holland, Amsterdam
- [27] Taylor, J.H., Fowler, L.A., and McCulloch, P.M. (1979): Nature, 277, 437
- [28] Treves, F. (1967): Topological Vector Spaces, Distributions and Kernels, Academic Press, New York
- [29] Tulczyjew, W. (1957): Bull. Acad. Polon. Sci. Classe III, 5, 279

LEBENS LAUF

- 27.01.51 geboren in Eichstätt/Bay. als
Sohn von Karljosef Schattner und
Irmingard Schattner, geb. Ried
- 1957/70 Schulbesuch in Eichstätt
- 20.06.70 Abitur am Willibald-Gymnasium in
Eichstätt
- 1970/71 Studium der Chemie an der Univer-
sität München
- 1971/74 und Studium der Physik an der Univer-
sität München
- 1975/79 Studium der Mathematik an der Uni-
versität Cambridge, England
- 1974/75
- seit 2.08.75 verheiratet mit Frédérique Vincendeau
- 3.08.77 Diplomprüfung in Physik, Diplomar-
beit bei Prof. Dr. J. Ehlers über
das Thema: "Der Schwerpunktsbegriff
in der Allgemeinen Relativitätstheorie"
- 15.09.77 - Promotionsstipendium der MPG, am MPI
für Physik und Astrophysik, München
- 14.09.79
- 25.01.78 Geburt meines Sohnes Hadrian
- seit 15.09.79 wissenschaftlicher Angestellter am
MPI für Astrophysik, Garching bei
München

